SOME DEVELOPMENT ON STUDIES OF UNIQUENESS AND STABILITY FOR INVERSE PROBLEMS FOR PARABOLIC, HYPERBOLIC AND SCHRÖDINGER EQUATIONS

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Dedicated to the 85th birthday of Professor Dr Vladimir Gavrilovich Romanov

As a principal inverse problem, we can refer to the determination of spatially varying coefficients for evolutionary partial differential equations by single observation data on subboundary. The mathematical issues are the uniqueness and the stability, and since a pioneering work Bukhgeim and Klibanov [3], such researches have been developed and now many results are available. Here we refer only to Bellassouend and Yamamoto [2], Isakov [4], Klibanov and Timonov [5], Yamamoto [7], [8].

However, the uniqueness and the stability are open in several important cases. The main purpose of this talk is to give affirmative answers to some of such open problems.

Let $\Omega \subset \mathbb{R}^d$ be a bounded smooth domain, $x = (x_1, ..., x_d) \in \mathbb{R}^d$, and ν be the unit outward normal vector to $\partial\Omega$. Moreover let $\gamma \subset \partial\Omega$ be an arbitrarily chosen subboundary, $0 \le t_0 \le T$ be arbitrarily fixed.

(I) Inverse parabolic problems with initial or final value problems

For $\partial_t u(x,t) = \Delta u(x,t) + p(x)u(x,t)$ in $\Omega \times (0,T)$, we consider the determination of $p(x)$, $x \in \Omega$ by data

$$(u|_{\gamma\times(0,T)},\ \nabla u|_{\gamma\times(0,T)},\ u(\cdot,t_0)|_\Omega).$$

Only for the case of $0 < t_0 < T$, the uniqueness and the stability are proved (e.g., [4], [7], [8]). The problems are not solved for $t_0 = 0$ and $t_0 = T$, in general.

We obtained

- the uniqueness for the one-dimensional case $\Omega := (0,\ell)$: Assuming that $\partial_x u(0,t) = 0$ for $0 < t < T$, we prove the uniqueness in determining $p(x)$, $0 < x < \ell$ only by data $(u(0,t),\ u(x,0))$ with $0 < t < T$ and $x \in (0,\ell)$. We stress that we have no data at another end $x = \ell$. This was an open problem even for the one-dimensional case. Moreover we describe a general scheme for establishing the uniqueness which is based on transformation operator (e.g., Levitan [6]) and the uniqueness for the inverse hyperbolic problem by Carleman estimate.

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- the uniqueness by data $(u|_{\gamma\times(0,T)},\ (\nabla u\cdot\nu)|_{\partial\Omega\times(0,T)},\ u(\cdot,0)|_\Omega)$, provided that the initial value $u(\cdot,0)$ is sufficiently smooth.
- the Lipschitz stability by data $(u|_{\gamma\times(0,T)},\ (\nabla u\cdot\nu)|_{\partial\Omega\times(0,T)},\ u(\cdot,T)|_\Omega)$.

(II) Sharp unique continuation for the Schrödinger equation

Let $\gamma\subset\partial\Omega$ and $T>0$ be arbitrarily chosen. Then, for $\sqrt{-1}\partial_t u+\Delta u=p(x)u$ in $\Omega\times(0,T)$, we show that if $u=\partial_\nu u=0$ on $\gamma\times(0,T)$, then $u=0$ in $\Omega\times(0,T)$. Moreover we apply it to inverse source problems.

(III) Inverse problems for transmission hyperbolic equations.

We consider a transmission equation where the wave speed is piecewise continuous and a source term in the form of $f(x)R(x,t)$ is attached. For suitably given $R(x,t)$, we are concerned with an inverse problem of determining $f(x)$ by initial values and Cauchy data on a suitable lateral subboundary. We prove the uniqueness and the stability for this inverse problem, which improves the results in Baudouin, Mercado and Osses [1]. The method relies on a Carleman estimate (Yamamoto [8]) which can be directly derived for hyperbolic equations of variable principal terms.

The contents of this talk are joint articles with Professor Oleg Y. Imanuvilov (Colorado State University).

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