Collision of Breathers on Surface of Deep Water

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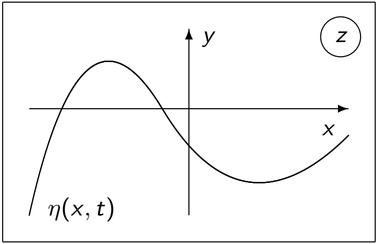
Gravity waves on the surface of deep water



Potential irrotational flow

$$\triangle\phi(x,y,t)=0$$

Boundary condition: $\left[\begin{array}{l} \frac{\partial\phi}{\partial t}+\frac{1}{2}|\nabla\phi|^2+g\eta=\frac{P}{\rho}, \\ \frac{\partial\eta}{\partial t}+\eta_x\phi_x=\phi_y \end{array}\right|$ at $y=\eta(x,t)$.

$$\frac{\partial\phi}{\partial y} = 0,\, y\to-\infty$$

Hamiltonian

Hamiltonian H is the total energy of fluid $H = T + U$

$$\begin{aligned} T &= \frac{1}{2}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\eta} (\nabla\Phi)^2 dy\,, \\ U &= \frac{g}{2}\int \eta^2 dx\,. \end{aligned}$$

$$\begin{aligned} \frac{\partial\eta}{\partial t} &= \frac{\delta H}{\delta\Psi}\,, \\ \frac{\partial\Psi}{\partial t} &= -\frac{\delta H}{\delta\eta}\,, \\ \Psi(x,t) &= \Phi(x,y,t)|_{y=\eta} \end{aligned}$$

$$H = \frac{1}{2}\int_{-\infty}^{\infty} \Psi\hat{G}(\eta)\Psi dx \quad + \quad \frac{g}{2}\int_{-\infty}^{\infty} \eta^2 dx$$

$$\begin{aligned} H = \quad & \frac{1}{2}\int g\eta^2 + \psi\hat{k}\psi dx - \frac{1}{2}\int \{(\hat{k}\psi)^2 - (\psi_x)^2\}\eta dx + \\ + \quad & \frac{1}{2}\int \{\psi_{xx}\eta^2\hat{k}\psi + \psi\hat{k}(\eta\hat{k}(\eta\hat{k}\psi))\} dx \end{aligned}$$

Normal variables b_k

- Consider waves moving in the same direction $k > 0$
- Nontrivial 4-wave resonance is absent. So, only wave scattering to itself is important

$$k + k_1 = k_2 + k_3, \qquad \omega_k + \omega_{k_1} = \omega_{k_2} + \omega_{k_3},$$

Hamiltonian:

$$\mathcal{H} = \int \omega_k b_k b_k^* dk + \frac{1}{2} \int T_{kk_1}^{k_2 k_3} b_k^* b_{k_1}^* b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} dk dk_1 dk_2 dk_3 + \ldots;$$

Corresponding dynamical equation is

$$i\dot{b} = \omega_k b_k + \frac{1}{2} \int T_{kk_1}^{k_2 k_3} b_{k_1}^* b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} dk_1 dk_2 dk_3$$

where

$$T_{k_2 k_3}^{k k_1} = \frac{\Theta(k)\Theta(k_1)\Theta(k_2)\Theta(k_3)}{8\pi} [k k_1 (k + k_1) + k_2 k_3 (k_2 + k_3) -$$
$$-(k k_2 |k - k_2| + k k_3 |k - k_3| + k_1 k_2 |k_1 - k_2| + k_1 k_3 |k_1 - k_3|)]$$

Hamiltonian and equation written in x-space

$$\mathcal{H} = \int b^* \hat{\omega}_k b dx + \frac{1}{2} \int |b'|^2 \left[\frac{i}{2}(bb'^* - b^* b') - \hat{K}|b|^2 \right] dx.$$

Corresponding dynamical equation is

$$i\dot{b} = \hat{\omega}_k b + \frac{i}{4}\hat{P^+} \left[b^* \frac{\partial}{\partial x}(b'^2) - \frac{\partial}{\partial x}(b^{*\prime} \frac{\partial}{\partial x} b^2) \right] -$$
$$-\frac{1}{2}\hat{P^+} \left[b \cdot \hat{K}(|b'|^2) - \frac{\partial}{\partial x}(b' \hat{K}(|b|^2)) \right].$$

$$kb_k^* \Leftrightarrow i\frac{\partial}{\partial x} b^*$$
$$kb_k \Leftrightarrow -i\frac{\partial}{\partial x} b$$
$$|k - k_2| b_k^* b_k \Leftrightarrow K(|b|^2)$$

A.I. Dyachenko, V.E. Zakharov A dynamical equation for water waves in one horizontal dimension, European Journal of Mechanics - B/Fluids, In Press, 32, (2012) p.17-21

Monochromatic wave

$$b(x) = B_0 e^{i(k_0 x - \omega_0 t)}$$

is the simplest solution. One can get the following relation:

$$\omega_0 = \omega_{k_0} + \frac{1}{2} k_0^3 |B_0|^2.$$

$$|B_0|^2 = \frac{\omega_{k_0}}{k_0} \eta_0^2,$$

One can recover well known Stokes correction to the frequency due to finite wave amplitude.

$$\omega_0 = \omega_{k_0}(1 + \frac{1}{2} k_0^2 |\eta_0|^2).$$

Modulational instability of monochromatic wave

Consider solution as follow:

$$b = (B_0 + \delta b(x,t))e^{i(k_0 x - \omega_0 t)}$$

where $B_0 = \text{const}$. Linearized equation for $b(x,t)$has solution as follow

$$\delta b \Rightarrow \delta b e^{\gamma_k t + i(kx - \Omega_k t)},$$

Then for growth rate γ_k the following formula is valid:

$$\gamma_k^2 = \frac{1}{8}\frac{\omega_{k_0}^2}{k_0^4}(1 - \mathbf{6}\mu^{\mathbf{2}})k^2\left[\mu^2(k_0 - \frac{|\mathbf{k}|}{\mathbf{2}})^2 - \frac{k^2}{8}\right].$$

Numerical simulations

$$i\dot{b} = \hat{\omega}_k b + \frac{i}{4}\hat{P^+}\left[b^*\frac{\partial}{\partial x}(b'^2) - \frac{\partial}{\partial x}(b^{*\prime}\frac{\partial}{\partial x}b^2)\right] - \\ -\frac{1}{2}\hat{P^+}\left[b\cdot\hat{K}(|b'|^2) - \frac{\partial}{\partial x}(b'\hat{K}(|b|^2))\right].$$

- Periodic boundary conditions $x \in [0, 2\pi]$
- Method Runge-Kutta $O(t^4)$
- FFTW (Fastest Fourier Transform in the West) library was used for fast Fourier Transform

Breather

Breather is the following solution:

$$b_k = e^{i(\Omega + Vk)t}\phi_k$$

where ϕ_k satisfies the equation:

$$(\Omega + Vk - \omega_k)\phi_k = \frac{1}{2}\int T_{kk_1}^{k_2 k_3}\phi_{k_1}^{*}\phi_{k_2}\phi_{k_3}\delta_{k+k_1-k_2-k_3} dk_1 dk_2 dk_3$$

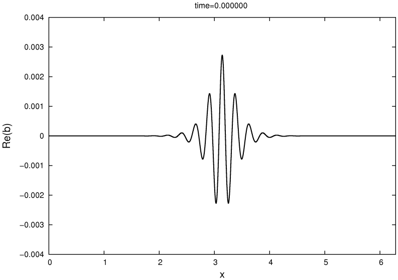
It can be found by Petviashvili method

$$\phi_k^{n+1} = \frac{NL_k^n}{M_k}\left[\frac{< \phi^n \cdot NL(\phi^n) >}{\phi^n \cdot M\phi^n}\right]^{\gamma}$$

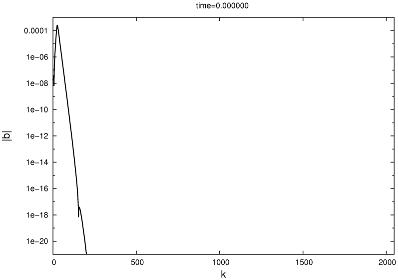
$$M_k = \Omega + Vk - \omega_k$$

$$NL(\phi^n) = \frac{i}{4}\left[\phi^{*}\frac{\partial}{\partial x}(\phi'^2) - \frac{\partial}{\partial x}(\phi^{*\prime}\frac{\partial}{\partial x}\phi^2)\right]^n - \frac{1}{2}\left[\phi \cdot \hat{K}(|\phi'|^2) - \frac{\partial}{\partial x}(\phi'\hat{K}(|\phi|^2))\right]^n$$

Results. Breather $\Omega = 2.53$, $V = 0.1$

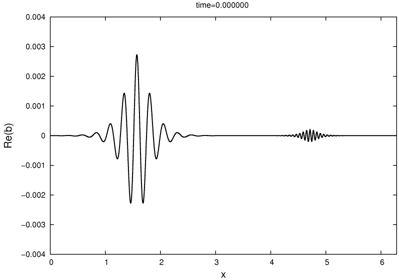


Real part of $b(x)$

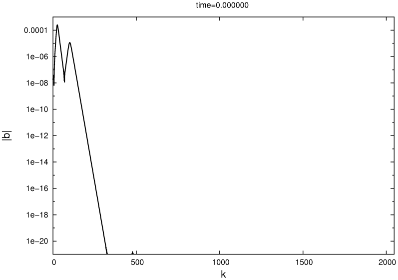


Spectrum of $b(x)$

Results. Collision of two breathers $\Omega_1 = 2.53 V_1 = 0.1, \Omega_2 = 5.1, V_1 = 0.05$



Real part of $b(x)$. Initial condition.



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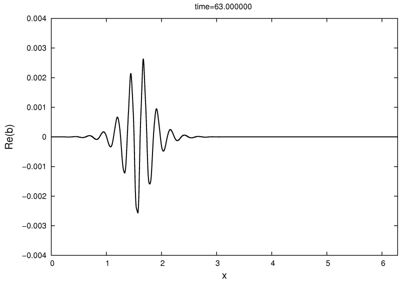
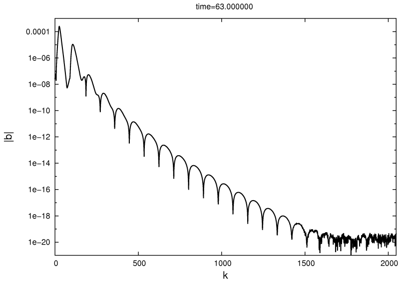


Real part of $b(x)$. Collision



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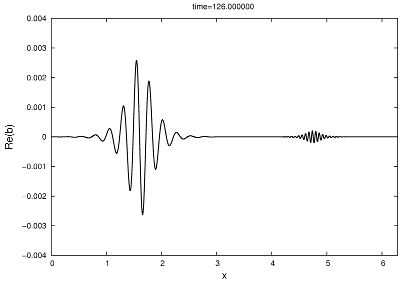
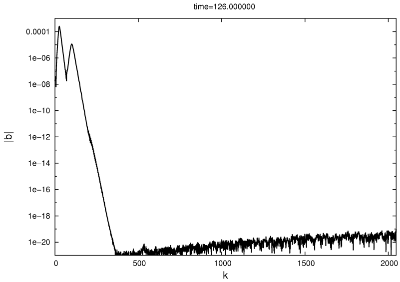


Real part $b(x)$. After collision



Spectrum of $b(x)$. After collision

Conclusions

- Localized in space breathers with different group velocities and amplitudes were found by iterative Petviashvili method. .
- Numerical simulations of collision of such breathers were conducted on the base of implemented program modelling Dyachenko-Zakharov equation.
- Numerical simulations of breathers collisions point on integrability of 2-D hydrodynamics with a free surface(for waves moving in the same direction)