On Nonuniqueness of Cycles in Dissipative Dynamical Systems of Chemical Kinetics

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We study periodic trajectories in odd-dimensional nonlinear dynamical systems of chemical kinetics considered as models of gene networks regulated by negative feedbacks. All equations here have the form

$$dx_1/dt=f_1(x_{2k+1})-x_{1.};\ldots dx_i/dt=f_i(x_{i-1})-x_i\,;\ldots dx_{2k+1}/dt=f_{2k+1}(x_{2k})-x_{2k+1} \qquad (1)$$

All variables have non-negative values and describe concentrations of species (proteins etc.). The functions f_j are positive and monotonically decreasing, this corresponds to negative feedbacks in gene networks, see [1].

Let $\boldsymbol{P}=[0,f_1(0)]\times\ldots\times[0,f_{2k+1}(0)]$. This is an invariant domain of the system (1). It was shown in [2], that phase portrait of each system of this type has unique stationary point X_0 in $\boldsymbol{P}$. Each bounded domain in these phase portraits collapses, its volume decreases exponentially in time. Let $\boldsymbol{M}$ be the matrix of linearization of the system (1) near the point X_0. This point is called hyperbolic, if the matrix $\boldsymbol{M}$ has eigenvalues with positive real parts, with negative real parts, and does not have imaginary eigenvalues.

Theorem 1. ([2]) If the point X_0 is hyperbolic, then there exists at least one cycle in the domain $\boldsymbol{P}$.

Sufficient analytic conditions (**SC**) of existence of a stable cycle C_1 in this domain were also obtained, [2].

We consider now questions of non-uniqueness of cycles in these dynamical systems. It follows from the Grobman-Hartman theorem that for any pair of complex conjugate eigenvalues of matrix $\boldsymbol{M}$ with positive real parts, there exist in some small neighbourhood U of X_0 a 2-dimensional unstable manifold S^2 invariant with respect to the flow of this system. Trajectories of all points of these manifolds are repelled from the point X_0 and remain in the parallelepiped $\boldsymbol{P}$. Numerical experiments show that if the real parts of corresponding eigenvalues are “sufficiently large”, then different unstable manifolds contain different cycles of the system (1), see [3].

If dimension of the system (1) is not a prime number: $2k+1 = p\cdot q$, where $p,q>1$, $p\neq q$, and the system (1) is symmetric (i.e., if all the functions f_i coincide, $f_i=f_0$), then its phase portrait contains two invariant planes S^p and S^q of dimensions p and q, respectively. (This is true even for arbitrary functions f_0, not necessary monotonic.) If X_0 is hyperbolic for restrictions of the system (1) on S^p, S^q, then Theorem 1 implies that each of these planes contains a cycle of this system. If conditions (**SC**) are satisfied, then each of S^p, S^q contains a cycle which is stable inside corresponding plane. Numerical experiments show (see for example Fig. 1) that this stability is not global, i.e., trajectories of the system (1) which start in "small" neighbourhoods of these planes near X_0 are attracted to a stable cycle C_1 of this system after some chaotic oscillations near S^p or S^q. These planes do not intersect C_1, so such a dynamical system has at least 3 different cycles. See [2,3], where the cases $2k+1 = 7, 9, 11, 13$ are studied as well.

Projections of trajectories of 15-dimensional system (1) on Fig. 1 below start on the left parts of the pictures near projections of the point X_0, make chaotic oscillations in the centres of these pictures, and then tend to projections of the cycle C_1 in the right parts. The planes of projections are spanned on eigenvectors of $\boldsymbol{M}$, not on standard basic vectors of the phase space $\boldsymbol{R}^{2k+1}$.

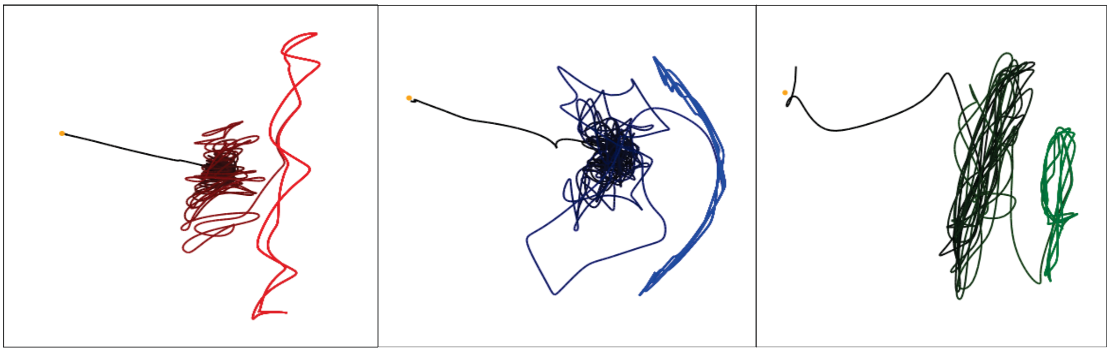

Fig. 1 Projections of different trajectories of 15-dimensional system (1) onto different 2-dimensional planes.

References

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