Verified Integration of ODEs with Taylor Models

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Keywords: ODEs, initial value problems, Taylor models

Verified integration methods for ODEs are methods that compute rigorous bounds for some specific solution or for the flow of some initial set of a given ODE. For almost fifty years, interval arithmetic has been used for calculating bounds for solutions of initial value problems. The origin of these methods dates back to Moore [5]. The most well-known interval method is the QR method due to Lohner [2], implemented in the AWA software package.

Unfortunately, interval methods sometimes suffer from overestimation. Pessimistic bounds are caused by the dependency problem, that is the lack of interval arithmetic to identify different occurrences of the same variable, and by the wrapping effect, which occurs when intermediate results of a calculation are enclosed into intervals.

Overestimation is a particular concern in the verified solution of initial value problems for ODEs. While it may sometimes be possible to reduce dependency by skillful reformulation of the given equations or by evaluating all function expressions by centered forms, the wrapping effect is more difficult to prevent. Interval methods usually compute enclosures of the flow at intermediate time steps of the integration domain. When the flow is a nonconvex set and is bounded by some convex interval, overestimation is inevitable.

For improving bounds, Taylor models have been developed as a combination of symbolic and interval computations by Berz and his group since the 1990s. In Taylor model methods, the basic data type is not a single interval, but a Taylor model $\mathcal{U} := p_n + \mathrm{i}$ consisting of a multivariate polynomial $\boldsymbol{p_n}$ of order $\boldsymbol{n}$ and some remainder interval i. In computations that involve $\boldsymbol{\mathcal{U}}$, the polynomial part is propagated by symbolic calculations where possible, and is thus hardly affected by the dependency problem or the wrapping effect. Only the interval remainder term and polynomial terms of order higher than $\boldsymbol{n}$, which are usually small, are bounded using interval arithmetic.

Besides reducing dependency, Taylor model methods for ODEs also benefit from their capability to represent non-convex sets. This is an intrinsic advantage over interval methods for enclosing the flows of nonlinear ODEs, especially in combination with large initial sets or with large integration domains [1, 3, 4, 6].

In our talk, we analyze Taylor model methods for the verified integration of ODEs and compare these methods with interval methods.

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