Interval arithmetic over finitely many endpoints

S.M. Rump, Hamburg/Tokyo

Summary:

Let a finite set $\mathbb{B}$ of interval *bounds* be given.

Which properties of $\mathbb{B}$ are necessary (and sufficient) such that an interval arithmetic over $\mathbb{B}$ satisfies as many as possible mathematical properties?

The goal:

For intervals A, B the following should be true without exception flag:

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&0 \in A - B \quad \Leftrightarrow \quad A \cap B \neq \emptyset \\
&0 \in A \cdot B \quad \Leftrightarrow \quad 0 \in A \cup B \\
&A \subseteq B/(B/A) \quad \text{if } 0 \notin A \cup B
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$$[\alpha, \beta] = \text{hull}(\text{interval}(\alpha), \text{interval}(\beta))$$

or $\quad A \subseteq \log(\exp(A)) \quad$ for any A $\qquad$ without exception flag

for finitely many interval bounds .

A standard definition of interval arithmetic

Start with *real bounds* $a, b, c, d \in \mathbb{R}$ and define

$$[a, b] + [c, d] = [a + c, b + d]$$
$$[a, b] \cdot [c, d] = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)]$$

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and continue with *floating-point bounds* $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in \mathbb{F}$:

$$[\tilde{a}, \tilde{b}] + [\tilde{c}, \tilde{d}] = [\nabla(\tilde{a} + \tilde{c}), \Delta(\tilde{b} + \tilde{d})]$$
etc.

Treatment of overflow

To cover overflow, an extension $\nabla, \Delta : \mathbb{R} \to \mathbb{F}^*$

with $\mathbb{F}^* := \mathbb{F} \cup \{-\infty, \infty\}$ is mandatory ($\Rightarrow$ *exception-free*).

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So far, so good.

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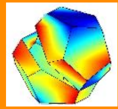

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Since $x \subseteq \mathbb{R}$ for all intervals x, it seems natural to define

$$\text{interval}(\infty) := \emptyset\ .$$

Unexpected, wrong results I

Consider

$$f(x) = \frac{10x + 5}{(e^x)^3} - 1 \ .$$

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function yy = cube(xx)
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without error message! But ...

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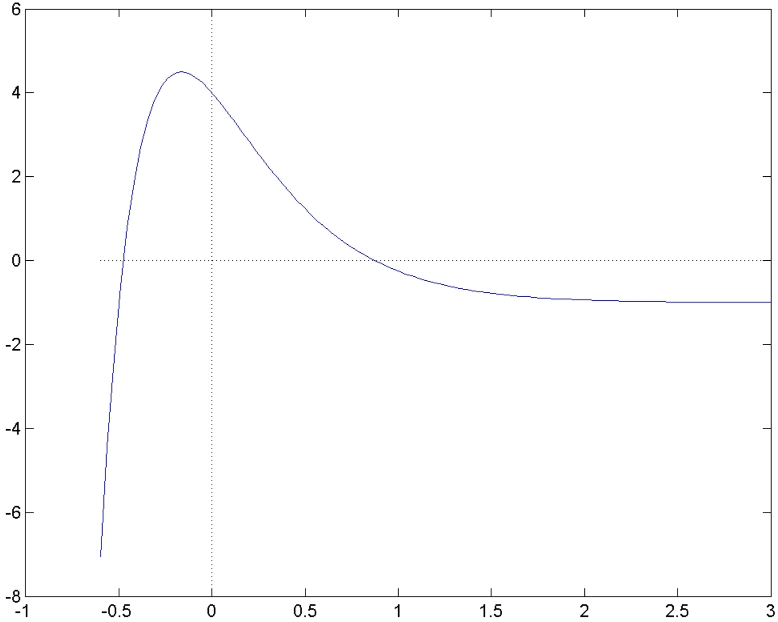
Unexpected, wrong results II

... there is a positive root: Graph of f between -0.6 and 3



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hence

$$\left(e^{[0,1000]}\right)^3 \subseteq \texttt{cube(exp(nums2interval(0,1000)))} = [1, 1],$$

a fatal mistake.

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And he admits:

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However, debugging requires a suspicion
(in the example `f(nums2interval(0,1000))` $\subseteq [4, 10004]$).

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Moreover, $\nabla(r) = -\Delta(-r)$.

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The natural definition `num2interval(r)`$= [\nabla(r), \Delta(r)]$ for $r \in \mathbb{R}^*$ implies

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Moreover, a best possible *real* interval $[r_1, r_2]$ is rounded

into the best possible *floating-point* interval $[\nabla(r_1), \Delta(r_2)]$,

which may serve to define all interval operations including functions.

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3) Not necessarily $\inf(xx), \sup(xx) \in xx$ for intervals xx.

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[An exception is infeasibility in optimization, please ask later.]

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Rather than just defining some new rounding or interval arithmetic,

we aim on a mathematical foundation.

Interval arithmetic over finitely many bounds

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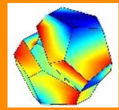

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Finally $\overline{\mathbb{IB}} = \mathbb{IB} \cup \{\mathtt{NaI}\}$; $A/B = \mathtt{NaI}$ for $0 \in B$.

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But $(-5)/3 \to \diamond(-5)/\diamond(3) = N/P_0 = \texttt{NaI}$ and $(-5)/3 \notin \diamond(-5)/\diamond(3)$.

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Note that division is excluded. Problem: $0 \in \diamond(\beta)$ for $\beta \neq 0$.

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Reference:

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