New Directions in Interval Linear Programming

Milan Hladík

Department of Applied Mathematics,
Faculty of Mathematics and Physics,
Charles University in Prague,
Czech Republic

SCAN 2012, Novosibirsk
September 23–29

Objectives

Objectives of the presentation

To show that interval linear programming

- has important applications
- has many nice results
- has challenging problems

Outline

1. Interval linear programming introduction
   - interval linear inequalities
   - complexity issues
2. Interval linear programming problems
   - optimal value range
   - optimal solution set
3. Interval linear programming applications
   - interval linear regression
   - constraint programming and global optimization

Interval linear inequalities

Notation

An interval matrix

$$\mathbf{A} := [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m\times n} \mid \underline{A} \le A \le \overline{A}\}.$$

The midpoint and radius matrices

$$A_c := \frac{1}{2}(\overline{A} + \underline{A}), \quad A_\Delta := \frac{1}{2}(\overline{A} - \underline{A}).$$

Theorem (Oettli–Prager, 1964)

A vector $x \in \mathbb{R}^n$ is a solution of $\mathbf{A}x = \mathbf{b}$ if and only if

$$|A_c x - b_c| \le A_\Delta |x| + b_\Delta.$$

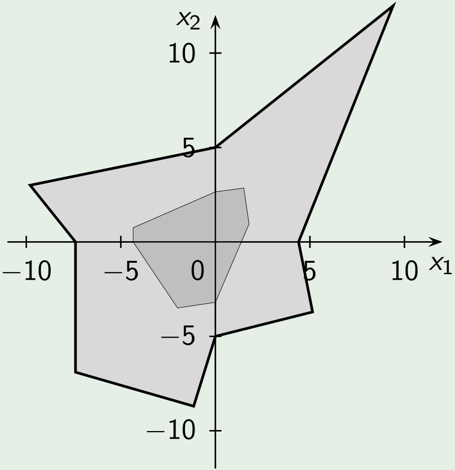
Theorem (Gerlach, 1981)

A vector $x \in \mathbb{R}^n$ is a solution of $\mathbf{A}x \le \mathbf{b}$ if and only if

$$A_c x - b_c \le A_\Delta |x| + b_\Delta.$$

Interval linear inequalities

Example (An interval polyhedron)



$$\begin{pmatrix} -[2,5] & -[7,11] \\ [1,13] & -[4,6] \\ [5,8] & [-2,1] \\ -[1,4] & [5,9] \\ -[5,6] & -[0,4] \end{pmatrix} x \leq \begin{pmatrix} [61,63] \\ [19,20] \\ [15,22] \\ [24,25] \\ [26,37] \end{pmatrix}$$

- union of all feasible sets in light gray,
- intersection of all feasible sets in dark gray,

Interval linear programming

Linear programming

Three basic forms of linear programs

$$f(A,b,c) \equiv \min c^T x \ \text{ subject to } \ Ax = b, \ x \geq 0,$$
$$f(A,b,c) \equiv \min c^T x \ \text{ subject to } \ Ax \leq b,$$
$$f(A,b,c) \equiv \min c^T x \ \text{ subject to } \ Ax \leq b, \ x \geq 0.$$

Interval linear programming

Family of linear programs with $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$, in short

$$f(\mathbf{A},\mathbf{b},\mathbf{c}) \equiv \min \mathbf{c}^T x \ \text{ subject to } \ \mathbf{A}x \overset{(\leq)}{=} \mathbf{b}, \ (x \geq 0).$$

The three forms are not transformable between each other!

Goals

- determine the optimal value range;
- determine a tight enclosure to the optimal solution set.

Complexity of basic problems

	$\mathbf{A}x = \mathbf{b},\ x \geq 0$	$\mathbf{A}x \leq \mathbf{b}$	$\mathbf{A}x \leq \mathbf{b},\ x \geq 0$
strong feasibility	co-NP-hard	polynomial	polynomial
weak feasibility	polynomial	NP-hard	polynomial
strong unboundedness	co-NP-hard	polynomial	polynomial
weak unboundedness	suff. / necessary conditions only	suff. / necessary conditions only	polynomial
strong optimality	co-NP-hard	co-NP-hard	polynomial
weak optimality	suff. / necessary conditions only	suff. / necessary conditions only	suff. / necessary conditions only
optimal value range	$\underline{f}$ polynomial $\overline{f}$ NP-hard	$\underline{f}$ NP-hard $\overline{f}$ polynomial	polynomial

Optimal value range

Optimal value range

Definition

$$\underline{f} := \min f(A,b,c) \ \text{ subject to } \ A \in \mathbf{A},\ b \in \mathbf{b},\ c \in \mathbf{c},$$
$$\overline{f} := \max f(A,b,c) \ \text{ subject to } \ A \in \mathbf{A},\ b \in \mathbf{b},\ c \in \mathbf{c}.$$

Theorem (Rohn, 2006)

We have for type ($\mathbf{A}x = \mathbf{b},\ x \geq 0$)

$$\underline{f} = \min \underline{c}^T x \ \textit{ subject to } \ \underline{A}x \leq \overline{b},\ \overline{A}x \geq \underline{b},\ x \geq 0,$$
$$\overline{f} = \max_{p \in \{\pm 1\}^m} f(A_c - \mathrm{diag}(p)\, A_\Delta, b_c + \mathrm{diag}(p)\, b_\Delta, \overline{c}).$$

Theorem (Vajda, 1961)

We have for type ($\mathbf{A}x \leq \mathbf{b},\ x \geq 0$)

$$\underline{f} = \min \underline{c}^T x \ \textit{ subject to } \ \underline{A}x \leq \overline{b},\ x \geq 0,$$
$$\overline{f} = \min \overline{c}^T x \ \textit{ subject to } \ \overline{A}x \leq \underline{b},\ x \geq 0.$$

Optimal value range

Algorithm (Optimal value range $[\underline{f}, \overline{f}]$)

1. Compute
$$\underline{f} := \inf\ c_c^T x - c_\Delta^T |x| \ \text{ subject to } \ x \in \mathcal{M},$$
where $\mathcal{M}$ is the primal solution set.
2. If $\underline{f} = \infty$, then set $\overline{f} := \infty$ and stop.
3. Compute
$$\overline{\varphi} := \sup\ b_c^T y + b_\Delta^T |y| \ \text{ subject to } \ y \in \mathcal{N},$$
where $\mathcal{N}$ is the dual solution set.
4. If $\overline{\varphi} = \infty$, then set $\overline{f} := \infty$ and stop.
5. If the primal problem is strongly feasible, then set $\overline{f} := \overline{\varphi}$; otherwise set $\overline{f} := \infty$.

Optimal solution set

Optimal solution set

The optimal solution set

Denote by $\mathcal{S}(A, b, c)$ the set of optimal solutions to

$$\min c^T x \;\text{ subject to }\; Ax = b,\; x \geq 0,$$

Then the optimal solution set is defined

$$\mathcal{S} := \bigcup_{A \in \mathbf{A},\, b \in \mathbf{b},\, c \in \mathbf{c}} \mathcal{S}(A, b, c).$$

Goal

Find a tight enclosure to $\mathcal{S}$.

Optimal solution set

Characterization

By duality theory, we have that $x \in \mathcal{S}$ if and only if there is some $y \in \mathbb{R}^m$, $A \in \mathbf{A}$, $b \in \mathbf{b}$, and $c \in \mathbf{c}$ such that

$$Ax = b,\ x \geq 0,\ A^T y \leq c,\ c^T x = b^T y,$$

where $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$.

Relaxation

Relaxing the dependencies

$$\mathbf{A}x = \mathbf{b},\ x \geq 0,\ \mathbf{A}^T y \leq \mathbf{c},\ \mathbf{c}^T x = \mathbf{b}^T y,$$

which is described by

$$\underline{A}x \leq \overline{b},\quad -\overline{A}x \leq -\underline{b},\quad x \geq 0,$$
$$A_c^T y - A_\Delta^T |y| \leq \overline{c},\quad |c_c^T x - b_c^T y| \leq c_\Delta^T x + b_\Delta^T |y|.$$

Linearization of $|y|$

Properties

- The solution set is non-convex in general
- It is linear at any orthant
- NP-hard to obtain exact bounds

Theorem (Beaumont, 1998)

For every $y \in \mathbf{y} \subset \mathbb{R}$ with $\underline{y} < \overline{y}$ one has

$$|y| \leq \alpha y + \beta, \tag{1}$$

where

$$\alpha = \frac{|\overline{y}| - |\underline{y}|}{\overline{y} - \underline{y}} \quad \text{and} \quad \beta = \frac{\overline{y}|\underline{y}| - \underline{y}|\overline{y}|}{\overline{y} - \underline{y}}.$$

Moreover, if $\underline{y} \geq 0$ or $\overline{y} \leq 0$ then (1) holds as equation.

Linearization of $|y|$

Now, the linearization reads

$$
\begin{aligned}
\underline{A}x \le \overline{b},\ -\overline{A}x \le -\underline{b},\ x \ge 0,\\
\big(A_c^T - A_\Delta^T \operatorname{diag}(\alpha)\big)y \le \overline{c} + A_\Delta^T\beta,\\
\underline{c}^T x + \big(-b_c^T - b_\Delta^T \operatorname{diag}(\alpha)\big)y \le b_\Delta^T\beta,\\
-\overline{c}^T x + \big(b_c^T - b_\Delta^T \operatorname{diag}(\alpha)\big)y \le b_\Delta^T\beta,
\end{aligned}
$$

where

$$
\alpha_i := \begin{cases} \frac{|\overline{y}_i| - |\underline{y}_i|}{\overline{y}_i - \underline{y}_i} & \text{if } \ \underline{y}_i < \overline{y}_i,\\ \operatorname{sgn}(\overline{y}_i) & \text{if } \ \underline{y}_i = \overline{y}_i, \end{cases}
$$

$$
\beta_i := \begin{cases} \frac{\overline{y}_i|\underline{y}_i| - \underline{y}_i|\overline{y}_i|}{\overline{y}_i - \underline{y}_i} & \text{if } \ \underline{y}_i < \overline{y}_i,\\ 0 & \text{if } \ \underline{y}_i = \overline{y}_i. \end{cases}
$$

Contractor

Algorithm (Optimal solution set contractor)

1. Compute an initial interval enclosure $\mathbf{x}^0, \mathbf{y}^0$
2. $i := 0$;
3. **repeat**
    1. compute the interval hull $\mathbf{x}^i, \mathbf{y}^i$ of the linearized system;
    2. $i := i + 1$;
4. **until** improvement is nonsignificant;
5. **return** $\mathbf{x}^i$;

Properties

- Each iteration requires computing the interval hull ($2(m+n)$ linear programs).
- In practice, it converges quickly, but not to $\mathcal{S}$ in general.

Example

Example

Consider an interval linear program

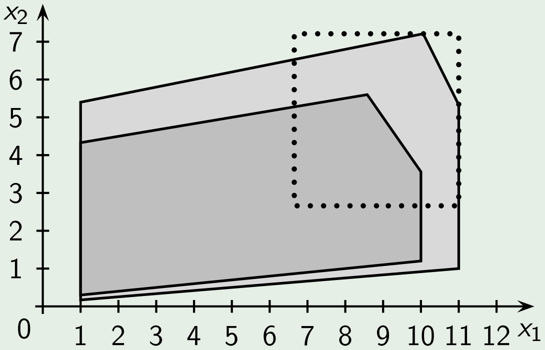
$$\begin{aligned} \min -[15,16]x_1 - [17,18]x_2 \ \text{ subject to}& \\ x_1 &\leq [10,11], \\ -x_1 + [5,6]x_2 &\leq [25,26], \\ [6,6.5]x_1 + [3,4.5]x_2 &\leq [81,82], \\ -x_1 &\leq -1, \\ x_1 - [10,12]x_2 &\leq -[1,2]. \end{aligned}$$

Take the initial enclosure

$$\mathbf{x}^0 = 1000 \cdot ([-1,1],\ [-1,1])^T,$$
$$\mathbf{y}^0 = 1000 \cdot ([0,1],\ [0,1],\ [0,1],\ [0,1],\ [0,1])^T.$$

Example

Example (cont.)



- Only four iterations needed.
- In grey the largest and the smallest feasible area.
- The final enclosure of the optimal solution set $\mathcal{S}$ is dotted.

Basis stability

Definition

The interval linear programming problem

$$\min \mathbf{c}^T x \quad \text{subject to} \quad \mathbf{A}x = \mathbf{b},\ x \geq 0,$$

is B-stable if B is an optimal basis for each realization.

Theorem

B-stability implies that the optimal value bounds are

$$\underline{f} = \min\ \underline{c}_B^T x \quad \textit{subject to} \quad \underline{A}_B x_B \leq \overline{b},\ -\overline{A}_B x_B \leq -\underline{b},\ x_B \geq 0,$$
$$\overline{f} = \max\ \overline{c}_B^T x \quad \textit{subject to} \quad \underline{A}_B x_B \leq \overline{b},\ -\overline{A}_B x_B \leq -\underline{b},\ x_B \geq 0.$$

Under the unique B-stability, the set of all optimal solutions reads

$$\underline{A}_B x_B \leq \overline{b},\ -\overline{A}_B x_B \leq -\underline{b},\ x_B \geq 0,\ x_N = 0.$$

Basis stability

Non-interval case

Basis B is optimal iff

C1. A_B is non-singular;

C2. $A_B^{-1}b \geq 0$;

C3. $c_N^T - c_B^T A_B^{-1} A_N \geq 0^T$.

Interval case

The problem is B-stable iff C1–C3 holds for each $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$.

Condition C1

- C1 says that $\mathbf{A}_B$ is regular;
- NP-hard problem;
- sufficient condition: $\rho\left(|((A_c)_B)^{-1}|(A_\Delta)_B\right) < 1$.

Basis stability

Non-interval case

Basis B is optimal iff

- C1. A_B is non-singular;
- C2. $A_B^{-1} b \geq 0$;
- C3. $c_N^T - c_B^T A_B^{-1} A_N \geq 0^T$.

Interval case

The problem is B-stable iff C1–C3 holds for each $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$.

Condition C2

- C2 says that the solution set to $\mathbf{A}_B x_B = \mathbf{b}$ lies in $\mathbb{R}^n_+$;
- sufficient condition: check of some enclosure to $\mathbf{A}_B x_B = \mathbf{b}$.

Basis stability

Non-interval case

Basis B is optimal iff

C1. A_B is non-singular;

C2. $A_B^{-1}b \geq 0$;

C3. $c_N^T - c_B^T A_B^{-1} A_N \geq 0^T$.

Interval case

The problem is B-stable iff C1–C3 holds for each $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$.

Condition C3

- C2 says that $\mathbf{A}_N^T y \leq \mathbf{c}_N,\ \mathbf{A}_B^T y = \mathbf{c}_B$ is strongly feasible;
- NP-hard problem;
- sufficient condition:
  $\overline{(\mathbf{A}_N^T)\mathbf{y}} \leq \underline{c}_N$, where $\mathbf{y}$ is an enclosure to $\mathbf{A}_B^T y = \mathbf{c}_B$.

Basis stability

Theorem

Condition C3 holds true if and only if for each $q \in \{\pm 1\}^m$ the polyhedral set described by

$$\begin{aligned} ((A_c)_B^T - (A_\Delta)_B^T \operatorname{diag}(q))y &\le \overline{c}_B, \\ -((A_c)_B^T + (A_\Delta)_B^T \operatorname{diag}(q))y &\le -\underline{c}_B, \\ \operatorname{diag}(q)\, y &\ge 0 \end{aligned}$$

lies inside the polyhedral set

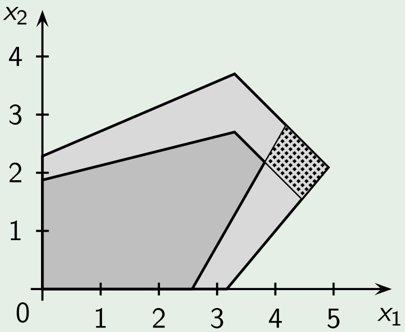
$$((A_c)_N^T + (A_\Delta)_N^T \operatorname{diag}(q))y \le \underline{c}_N, \ \operatorname{diag}(q)\, y \ge 0.$$

Basis stability

Example

Consider an interval linear program

$$\max\ ([5,6],[1,2])^T x \ \text{ s.t. } \begin{pmatrix} -[2,3] & [7,8] \\ [6,7] & -[4,5] \\ 1 & 1 \end{pmatrix} x \le \begin{pmatrix} [15,16] \\ [18,19] \\ [6,7] \end{pmatrix},\ x \ge 0.$$



- union of all feasible sets in light gray,
- intersection of all feasible sets in dark gray,
- set of optimal solutions in dotted area

Open problems

- A sufficient and necessary condition for weak unboundedness, strong boundedness and weak optimality.
- A method to check if a given $x^* \in \mathbb{R}^n$ is an optimal solution for some realization.
- A method for determining the image of the optimal value function.
- A sufficient and necessary condition for duality gap to be zero for each realization.
- A method to test if a basis B is optimal for some realization.
- Tight enclosure to the optimal solution set.

Applications

Applications

Applications

- real-life problems affected by uncertainties
  - economics (portfolio selection,...)
  - environmental management (water resource and waste mng. planning)
  - logistic
  - ...
- technical tool in constraint programming and global optimization
  technical tool in constraint programming and global optimization
- others
  - interval matrix games
  - interval linear regression interval linear regression
  - measure of sensitivity of linear programs

Interval linear regression

Linear regression

Consider a linear regression model

$$X\beta \approx y,$$

Find β solving

$$\min_{\beta\in\mathbb{R}^m} \|X\beta - y\|_p.$$

L_p-norm

- $\min_{\beta\in\mathbb{R}^m} \|X\beta - y\|_2$ ... least squares

$$\beta = (X^TX)^{-1}X^Ty,$$

- $\min_{\beta\in\mathbb{R}^m} \|X\beta - y\|_1$ ... least absolute deviations

$$\min e^Tw \ \text{ subject to } \ X\beta - y \le w,\ -X\beta + y \le w,\ w \ge 0.$$

- $\min_{\beta\in\mathbb{R}^m} \|X\beta - y\|_\infty$ ... Chebyshev approximation

$$\min t \ \text{ subject to } \ X\beta - y \le te,\ -X\beta + y \le te,\ t \ge 0,$$

Interval linear regression

Interval linear regression

Consider a system of linear regression models

$$X\beta \approx y,$$

where $X \in \mathbf{X}$ and $y \in \mathbf{y}$.

Reduction to Interval linear programming

- For L_1-norm and L_∞-norm, we get an interval linear program.
- Optimal value range ... minimal/maximal residual value
- Optimal solution set ... set of all regression parameters
- Illustration of basis stability:
  - interpret β as a classifier of data (X, y) to two classes *below* and *above* regression line
  - basis stability = the same classification for any realization

Constraint programming and global optimization

Constraint programming problem

Enclose the set $\mathcal{S}$ described by

$$f_i(x_1,\dots,x_n)=0,\quad i=1,\dots,m, \qquad (\,f(x)=0\,)$$
$$g_j(x_1,\dots,x_n)\le 0,\quad j=1,\dots,\ell, \qquad (\,g(x)\le 0\,)$$

on a box $\mathbf{x}$.

Global optimization problem

Find

$$\min \varphi(x)$$

subject to

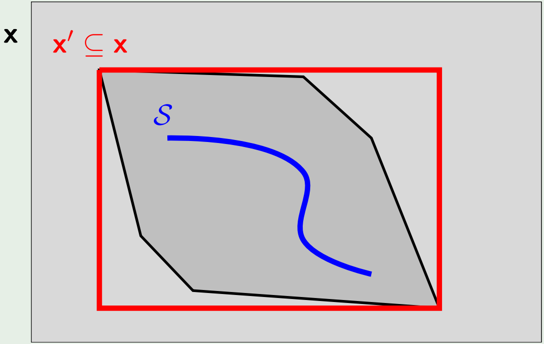
$$f(x)=0,\quad g(x)\le 0,\quad x\in\mathbf{x}.$$

Constraint programming and global optimization

Interval linear programming approach

- linearize constraints,
- compute new bounds and iterate.

Example



Constraint programming and global optimization

Interval linearization

Let $x^0 \in \mathbf{x}$. Suppose that a for some interval matrices $\mathbf{A}$ and $\mathbf{B}$ we have

$$f(x) \subseteq \mathbf{A}(x - x^0) + f(x^0), \quad \forall x \in \mathbf{x}$$
$$g(x) \subseteq \mathbf{B}(x - x^0) + g(x^0), \quad \forall x \in \mathbf{x},$$

e.g. by the mean value form, slopes, ...

Interval linear programming formulation

Now, the set $\mathcal{S}$ is enclosed by

$$\mathbf{A}(x - x^0) + f(x^0) = 0,$$
$$\mathbf{B}(x - x^0) + g(x^0) \leq 0.$$

What remains to do

- Solve the interval linear program
- choose $x^0 \in \mathbf{x}$

Case $x^0 := \underline{x}$

Let $x^0 := \underline{x}$. Since $x - \underline{x}$ is non-negative, the solution set to

$$\mathbf{A}(x - x^0) + f(x^0) = 0,$$
$$\mathbf{B}(x - x^0) + g(x^0) \le 0,$$

is described by

$$\underline{A}x \le \underline{A}\,\underline{x} - f(\underline{x}), \quad \overline{A}x \ge \overline{A}\underline{x} - f(\underline{x}),$$
$$\underline{B}x \le \underline{B}\,\underline{x} - g(\underline{x}).$$

- Similarly if x^0 is any other vertex of $\mathbf{x}$

General case

Let $x^0 \in \mathbf{x}$. The solution set to

$$\begin{aligned}\mathbf{A}(x - x^0) + f(x^0) &= 0,\\ \mathbf{B}(x - x^0) + g(x^0) &\le 0,\end{aligned}$$

is described by

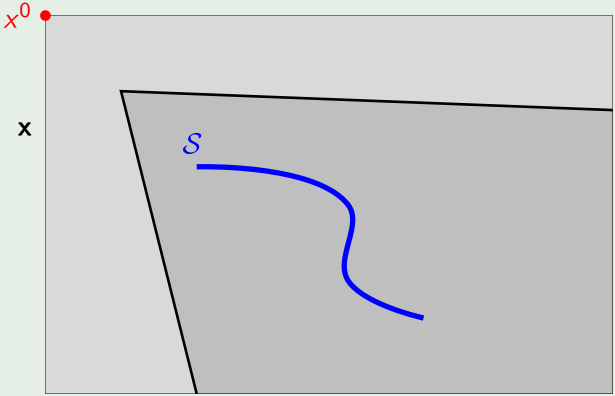
$$\begin{aligned}|A_c(x - x^0) + f(x^0)| &\le A_\Delta |x - x^0|,\\ B_c(x - x^0) + g(x^0) &\le B_\Delta |x - x^0|.\end{aligned}$$

- Non-linear description due to the absolute values.
- How to get rid of them?

Constraint programming and global optimization

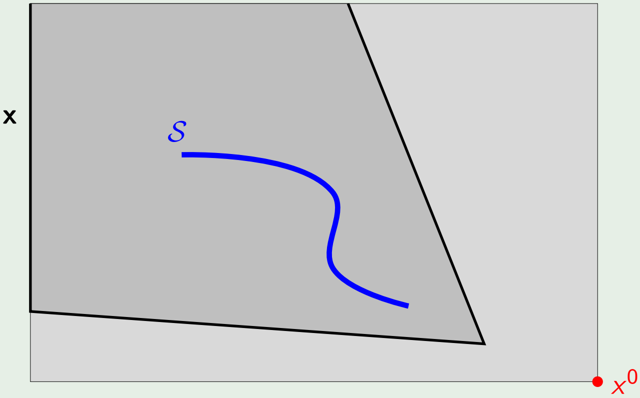
Example

Typical situation when choosing x^0 to be vertex:



Example

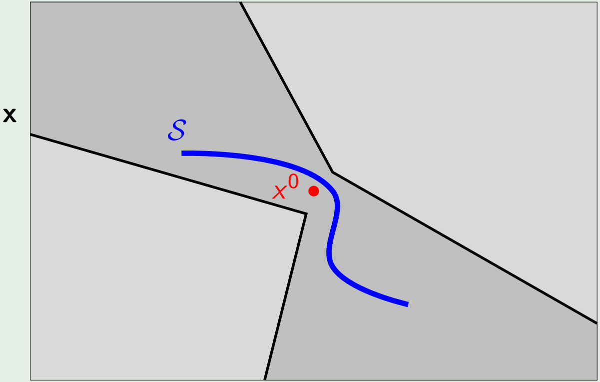
Typical situation when choosing x^0 to be the opposite vertex:



Constraint programming and global optimization

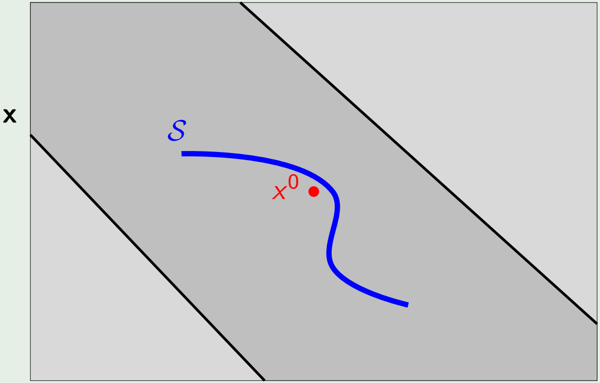
Example

Typical situation when choosing $x^0 = x_c$:



Example

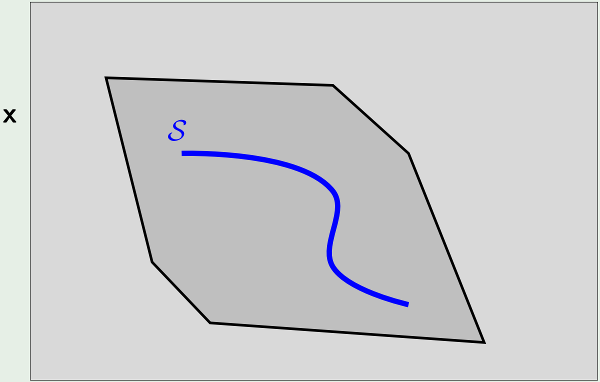
Typical situation when choosing $x^0 = x_c$ (after linearization):



Constraint programming and global optimization

Example

Typical situation when choosing all of them:



Conclusion

My apologies for not mentioning

- duality in interval linear programming
- linear programming verification
- fuzzy linear programming
- . . . and many others

Challenging problems

- enclose optimal solution set
- handle dependencies
- others (inner enclosures, . . .)