On Unboundedness of Generalized Solution Sets for Interval Linear Systems

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We consider systems of linear interval equations of the form

$$\mathbf{A}x = \mathbf{b}$$

where $\mathbf{A} = [\underline{A}, \overline{A}]$ **is an interval** $m \times n$**-matrix,** $\mathbf{b} = [\underline{b}, \overline{b}]$ **is an interval** m**-vector, and** $x \in \mathbb{R}^n$**. The interval matrix and the interval vector are traditionally understood as the sets**

$$\mathbf{A} = \{\, A \in \mathbb{R}^{m\times n} \mid \underline{A} \le A \le \overline{A} \,\}, \;\; \mathbf{b} = \{\, b \in \mathbb{R}^{m} \mid \underline{b} \le b \le \overline{b} \,\}$$

(by $\mathbb{R}^{m\times n}$ **from now on we denote the set of** $m \times n$**-matrices). It is also assumed that** $\underline{A} \le \overline{A}$**,** $\underline{b} \le \overline{b}$**, and the inequalities between the matrices and the vectors are understood elementwise and coordinatewise, respectively.**

Following the papers S.P.Shary [1], we suppose that an quantifier $m \times n$**-matrix** $\Lambda = (\lambda_{ij}), \lambda_{ij} \in \{-1, 1\}$**,** $i = \overline{1, m}$**,** $j = \overline{1, n}$ **and an quantifier** m**-vector** $\beta = (\beta_1, \ldots, \beta_n)^\top$**,** $\beta_i \in \{-1, 1\}$**,** $i = \overline{1, m}$ **are given along with the interval** $m \times n$**-matrix A and the interval** m**-vector b.**

The matrix $\mathbf{A} = (\mathbf{a}_{ij})$ **is decomposed into the two matrices** $\mathbf{A}^\exists = (\mathbf{a}_{ij}^\exists)$ **and** $\mathbf{A}^\forall = (\mathbf{a}_{ij}^\forall)$ **so that**

$$\mathbf{a}_{ij}^\exists = \begin{cases} \mathbf{a}_{ij}, & \textbf{if} \ \ \lambda_{ij} = 1, \\ 0, & \textbf{if} \ \ \lambda_{ij} = -1, \end{cases} \qquad \mathbf{a}_{ij}^\forall = \begin{cases} 0, & \textbf{if} \ \ \lambda_{ij} = 1, \\ \mathbf{a}_{ij}, & \textbf{if} \ \ \lambda_{ij} = -1. \end{cases}$$

Similarly, let us decompose the vector $\mathbf{b} = (\mathbf{b}_1, \ldots, \mathbf{b}_m)^\top$ **into the two vectors** $\mathbf{b}^\exists = (\mathbf{b}_1^\exists, \ldots, \mathbf{b}_m^\exists)^\top$ **and** $\mathbf{b}^\forall = (\mathbf{b}_1^\forall, \ldots, \mathbf{b}_m^\forall)^\top$ **such that**

$$\mathbf{b}_i^\exists = \begin{cases} \mathbf{b}_i, & \textbf{if} \ \ \beta_i = 1, \\ 0, & \textbf{if} \ \ \beta_i = -1, \end{cases} \qquad \mathbf{b}_i^\forall = \begin{cases} 0, & \textbf{if} \ \ \beta_i = 1, \\ \mathbf{b}_i, & \textbf{if} \ \ \beta_i = -1. \end{cases}$$

It is furthermore obvious that $\mathbf{A} = \mathbf{A}^\forall + \mathbf{A}^\exists$**,** $\mathbf{b} = \mathbf{b}^\forall + \mathbf{b}^\exists$**.**

Definition 1(S.P.Shary [1]). For given quantifier matrix Λ and quantifier vector β, the generalized AE-solution set of the type $\Lambda\beta$ is

$$\Xi_{\Lambda,\beta}(\mathbf{A},\mathbf{b}) = \Big\{ x \in \mathbb{R}^n \mid (\forall A' \in \mathbf{A}^{\forall})(\forall b' \in \mathbf{b}^{\forall}) \\ (\exists A'' \in \mathbf{A}^{\exists})(\exists b'' \in \mathbf{b}^{\exists})((A' + A'')\, x = b' + b'') \Big\}. \quad (1)$$

The main purpose of our paper is to inquire into the algorithmic complexity of the problem relating to these sets:

Problem. To find out (to determine) whether the set (1) is unbounded or not.

In the rest of the paper, for the two $m \times n$-matrices $A = (a_{ij})$ and $B = (b_{ij})$, by $A \circ B$ we will denote their Hadamard product $A \circ B = (a_{ij} b_{ij})$.

Using the well-known Oettli-Prager theorem, it is possible to obtain Oettli-Prager-type description of the generalized solution sets.

For any given Λ and β, the equality

$$\Xi_{\Lambda,\beta}(\mathbf{A},\mathbf{b}) = \{\, x \in \mathbb{R}^n \mid |A_c x - b_c| \le (\Lambda \circ \Delta)|x| + \beta \circ \delta \,\},$$

holds, where $A_c = \frac{1}{2}(\underline{A} + \overline{A})$**,** $\Delta = \frac{1}{2}(\overline{A} - \underline{A})$**,** $b_c = \frac{1}{2}(\underline{b} + \overline{b})$**,** $\delta = \frac{1}{2}(\overline{b} - \underline{b})$**.**

Using this description, we obtain the following statement.

Proposition. The set $\Xi_{\Lambda,\beta}(\mathbf{A},\mathbf{b})$ *is unbounded if and only if for some* $y \in Q = \{\, x \in \mathbb{R}^n \mid x_i \in \{-1, 1\},\ i = \overline{1, n}\}$ *there exists a solution to the following system of linear inequalities (where* $T_y = diag(y_1, \ldots, y_n)$*)*

$$\begin{cases} -(\Lambda \circ \Delta)T_y x - \beta \circ \delta \le A_c x - b_c \le (\Lambda \circ \Delta)T_y x + \beta \circ \delta,\ T_y x \ge 0, \\ -(\Lambda \circ \Delta)T_y z \le A_c z \le (\Lambda \circ \Delta)T_y z,\ T_y z \ge 0, \sum_{i=1}^{i=n} y_i z_i \ge 1. \end{cases} \quad (2)$$

In order to correctly state the problems of interest for us, we shall assume that for each m and n there are a fixed $m \times n$-matrix $\Lambda(m,n) = (\lambda_{ij}(m,n))$ such that $\lambda_{ij}(m,n) \in \{-1,1\}$, $i = \overline{1,m}$, $j = \overline{1,n}$ and an m-vector $\beta(m) = (\beta_1(m),\ldots,\beta_m(m))^T$ such that $\beta_i(m) \in \{-1,1\}$, $i = \overline{1,m}$.

In other words, the two functions Λ and β are given, such that the function Λ determines a correspondence between $m \times n$-matrices of $\{-1,1\}$ and the pairs of natural numbers (m,n), $(m \geq 1, n \geq 1)$, while the function β sets a correspondence between the m-vectors of $\{-1,1\}$ and the natural numbers m.

Hence, for any interval system of the form (1) having m equations for n variables, it is possible to define the set $\Xi_{\Lambda(m,n),\beta(m)}(\mathbf{A},\mathbf{b})$. Furthermore, let assume that the matrix $\Lambda(m,n)$ and the vector $\beta(m)$ are “easily computable” in the following sense.

Definition 2 Let us speak that the functions Λ and β are ***easily computable*** if there exists a pseudo-polynomial time algorithm computing the matrix $\Lambda(m, n)$ and the vector $\beta(m)$, i.e. the algorithm whose processing time is limited by the polynomial of m and n.

Also, we will say that an interval matrix $\mathbf{A}$ is ***integer*** if the endpoints of its entries are integer numbers.

Let us consider the following problem:

Problem $UNB(\Lambda, \beta)$
(checking unboundedness of the generalized solution sets $\Xi_{\Lambda,\beta}(\mathbf{A}, \mathbf{b})$)

Given. An integer interval $m \times n$-matrix $\mathbf{A} = [A_c - \Delta,\ A_c + \Delta]$ and an integer interval m-vector $\mathbf{b} = [b_c - \delta,\ b_c + \delta]$.

Question. Is it true that $\Xi_{\Lambda(m,n),\beta(m)}(\mathbf{A}, \mathbf{b})$ unboundedness?

It will be obvious from the foregoing considerations that the computational complexity of these problems is substantially determined by the number of the existential quantifiers in the definition of $\Xi_{\Lambda,\beta}(\mathbf{A}, \mathrm{b})$, i.e. by the number of $(+1)$'s in the matrix $\Lambda(m,n)$ and in the vector $\beta(m)$. Roughly speaking, if the number of existential quantifiers is "large enough", that is, a sufficiently large number of the columns of the matrix Λ contain at least one $(+1)$, and a sufficiently large number of rows of the extended matrix $(\Lambda\beta)$ contain at least one $+1$, then the above formulated problems is NP-complete. If the total number of $(+1)$'s in the matrix Λ grows slowly in comparison with the number mn (specifically, it has the order of $\log_2(mn)$), then these problems can be solved in the polynomial time.

To formulate what is meant by the term "sufficiently many existential quantifiers", we need giving additional clarification. When defining the term precisely, we will use usual notation for the submatrices of some matrix, i.e. if $\Lambda = (\lambda_{ij})$ **is an** $m \times n$**-matrix and** $I = \{i_1, \ldots, i_k\}$**,** $J = \{j_1, \ldots, j_l\}$**,** $(1 \leq i_1 < i_2 \ldots < i_k \leq m,\ 1 \leq j_1 < j_2 \ldots < j_l \leq n$**, then by** $\Lambda(I|J)$ **we denote the** $k \times l$**-matrix located at the intersections of the rows with the numbers** $i_1, \ldots, i_k$ **and the columns with the numbers** $j_1, \ldots, j_l$**. Similarly, for the** m**-vector** β **and** $I = \{i_1, \ldots, i_k\}$**,** $1 \leq i_1 < \ldots < i_k \leq m$**, by** $\beta(I)$ **we denote the** k**-vector with the corresponding coordinates.**

Definition 3 Let us say that the functions Λ and β are ***computationally 1-saturate*** (or, briefly, 1-saturate) if there exists an algorithm allowing the numbers m, n, k, l and the two submatrices Λ_0, Λ_1 of the matrix $\Lambda(m,n)$ of dimensions $k \times s$ and $s \times l$, respectively, to be found for any natural number s, so that the following conditions hold:

1) the running time of the algorithms is restricted by a polynomial of s (that is, similar to Definition 1, the algorithm is quasi-polynomial with respect to s);

2) $m \geq k + l + s + 1, \ \ n \geq l + s$;

3) if $\Lambda_0 = \Lambda(m,n)(K \mid J)$, $\Lambda_1 = \Lambda(m,n)(I \mid L)$, then $K \bigcap I = J \bigcap L = \emptyset$, i.e. submatrices are located in different rows and different columns;

4) each of the columns in the submatrix Λ_0 contains at least one $(+1)$;

5) each of the rows in the submatrix $(\Lambda_1\gamma)$ obtained by adding of the column $\gamma = \beta(m)(I)$ to the submatrix Λ contains at least one $(+1)$;

In other words, up to within the transposition of rows and columns, the extended matrix $(\Lambda\beta)$ has the form

$$(\Lambda(m,n)\beta(m)) = \begin{pmatrix} \Lambda_0 & * & * & * \\ * & \Lambda_1 & * & \gamma \\ * & * & * & * \end{pmatrix}, \tag{3}$$

where the submatrices Λ_0 and Λ_1 possess the properties 4) and 5) from the Definition 3.

Comment. Denote by $U_\Lambda(m,n)$ the number of $(+1)$'s in the the matrix $\Lambda(m,n)$. If the functions Λ, β are 1-saturate then from the Condition 1) of the Definition 3 and from the fact that the complexity of the matrix $\Lambda(m,n)$ is greater or equal to mn it follows that there exist such numbers $C > 1$, $M > 1$ that $mn \le Cs^M$.

Since $U_\Lambda(m, n) \geq s$ according to Condition 4, we get the estimate

$$U_\Lambda(m, n) \geq \left(\frac{1}{C}\right)^{\frac{1}{M}} (mn)^{\frac{1}{M}},$$

i.e., in this case for some $M > 1$ the relation

$$\limsup_{m,n\to\infty} \frac{U_\Lambda(m, n)}{\sqrt[M]{mn}} > 0. \tag{4}$$

holds true.

Therefore, the condition (4) is at least necessary for the functions Λ, β to be 1-saturate. It imposes a restriction from below on the order of growth of $U_\Lambda(m, n)$.

Theorem 1 **If the functions Λ, β are easily computable and 1-saturate then the Problem $UNB(\Lambda, \beta)$ is NP-complete.**

Let us now show that if the number of $(+1)$'s in the matrix $\Lambda(m,n)$ is "not too large", then the problem $UNB(\Lambda,\beta)$ is polynomially solvable.

Theorem 2 **If the functions Λ, β are easily computable and if the condition**

$$\limsup_{m,n\to\infty} \frac{U_{\Lambda}(m,n)}{\log_2(mn)} \le C$$

holds for some fixed integer C, then there exist polynomial time algorithms that solve the problem $UNB(\Lambda,\beta)$.

In particular, it follows from the theorem that if $P \neq NP$ then there exists no better criterion for unboundedness of the generalized set of solutions than checking solvability for 2^n systems of the form (2).

References:

[**1**] S.P. Shary, **A new technique in systems analysis under interval uncertainty and ambiguity,** *Reliable Computing*, **8 (2002), No. 5, pp. 321–418.**

[**2**] A.V. Lakeyev, **Computational Complexity of Estimation of Generalized Solution Sets for Interval Linear Systems,** *Computation Technologies*, **8 (2003), No. 1, pp. 12–23.**

THANK YOU FOR ATTENTION!