Calculation of potential and force of attraction of an ellipsoid

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The quadrature for integral from product of functions

- Let us consider the quadrature to calculate an integrals having the form

$$\int_a^b f(x)g(x)dx \qquad (1)$$

$$f(x) \in C[a,b], \qquad g(x) \in L^1[a,b], \qquad g(x) \geq 0 \;\; (g(x) \leq 0) \;\; when \;\; x \in [a,b]$$

in case that the integrals $G(\alpha,\beta) = \int_\alpha^\beta g(x) \;\; dx\,, \quad \alpha,\beta \in [a,b]$

can be calculated exactly.

Let the grid on interval $[a,b]$ is defined with nodes $x_i\,,\;\; i = 1,\ldots,n$

$$x_{i+\frac{1}{2}} = \frac{x_i + x_{i+1}}{2}, \quad h_i = x_{i+1} - x_i\,, \quad i = 1,\ldots,n-1.$$

Give the piecewise constant approximation of function $f(x)$ on interval $[a,b]$ by polynomial $L_0(x)$

$$L_0(x) = f(x_i) \;\; when \;\; x \in \left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right]\,,\; i = 2,\ldots,n-1,$$

$$L_0(x) = f(x_1) \;\; when \;\; x \in \left[a, x_{\frac{3}{2}}\right),$$

$$L_0(x) = f(x_n) \;\; when \;\; x \in \left(x_{n-\frac{1}{2}}, b\right].$$

Then for estimation of error $\varepsilon_i = \int_{x_{i-1/2}}^{x_{i+1/2}} [f(x) - L_0(x)] g(x) dx$

the following inequalities are valid

$$|\varepsilon_i| \le \frac{1}{2} \max_{x \in [x_{i-1/2}, x_{i+1/2}]} |f'(x)| \ \max\{h_{i-1}, h_i\} \ G\left(x_{i-1/2}, x_{i+1/2}\right), \ i = 2, \ldots, n-1 \quad ,$$

$$|\varepsilon_1| \le \max_{x \in [a, x_{3/2})} |f'(x)| \ \max\left\{x_1 - a, \ \frac{h_1}{2}\right\} \ G\left(a, x_{3/2}\right) \ ,$$

$$|\varepsilon_n| \le \max_{x \in (x_{n-1/2}, b]} |f'(x)| \ \max\left\{b - x_n, \ \frac{h_{n-1}}{2}\right\} \ G\left(x_{n-1/2}, b\right) \ .$$

Then for total error $\varepsilon = \int_a^b [f(x) - L_0(x)] g(x) dx$ the inequality $|\varepsilon| \le h_0 M_0 G(a, b)$,

is taken place, where $M_0 = \max_{x \in [a,b]} |f'(x)|, \ h_0 = \max\left\{x_1 - a, \ b - x_n, \ \frac{1}{2} \max_i \{h_i\}\right\}.$

Then the quadrature for approximation of integral (1) will be

$$\int_a^b f(x) g(x) \ dx \approx \int_a^b L_0(x) g(x) \ dx = \tag{2}$$

$$f(x_1) G\left(a, x_{3/2}\right) + f(x_n) G\left(x_{n-1/2}, b\right) + \sum_{i=2}^{n-1} f(x_i) G\left(x_{i-1/2}, x_{i+1/2}\right)$$

Calculation of potential of an ellipsoid

- Let us consider the body T with density ρ , bounded by surface of an ellipsoid. Choose the Cartesian coordinates with origin in centre of ellipsoid, with axes directed along the main axes of the ellipsoid. The potential of body in point $M_0 = M_0(x_0, y_0, z_0)$ is defined by formula

$$U(M_0) = \int_T \frac{\rho(M)}{|M - M_0|} \, d\tau \, .$$

Pass to spherical coordinates $\varphi, \ \theta, r$. Then

$$U(M_0) = \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\varphi \int_0^{R(\varphi,\theta)} \frac{r^2 \rho(\varphi, \theta, r)}{\sqrt{r_0^2 - 2rr_0 \cos\psi + r^2}} dr \qquad (3)$$

$$\cos\psi = \cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\varphi - \varphi_0)$$

Denote $f(r) = \rho(\varphi, \theta, r)$, $g(r) = \dfrac{r^2}{\sqrt{r_0^2 - 2rr_0\cos\psi + r^2}}$

for fixed values of variables $\varphi, \ \varphi_0, \ \theta, \ \theta_0$.

Calculating the integrals from the function $g(r)$, we obtain

$$G(r_i, r_{i+1}) = \int_{r_i}^{r_{i+1}} g(r)\,dr = G_0(r_{i+1}) - G_0(r_i),$$

where $G_0(x) = \frac{1}{2}\Big[(x + 3r_0\cos\psi)\,sq(x) + r_0^2\left(3\cos^2\psi - 1\right)\ \ln\big(w(x)\big)\Big],$

$$sq(x) = \sqrt{r_0^2 - 2r_0 x\cos\psi + x^2}\ , \qquad w(x) = sq(x) + x - r_0\cos\psi\ .$$

To calculate the complete interior integral in (3) , we will use the quadrature (2).
A weak logarithmic singularity in summands $G_0(r_i)$ when $\psi = 0$ and $r_i = r_0$ can be taken into account in numerical integration by introducing a new variable $\varphi - \varphi_0 = t^3$.

Calculation of force of attraction of an ellipsoid

- The force of attraction in spherical coordinates is the vector

$$\mathbf{F} = \left(\frac{\partial U}{\partial \varphi_0}, \frac{\partial U}{\partial \theta_0}, \frac{\partial U}{\partial r_0} \right)^T$$

3.1 The component of force of attraction by coordinate r_0

Denote $f(r) = \rho(\varphi, \theta, r)$, $g(r) = \frac{\partial}{\partial r_0} \frac{r^2}{\sqrt{r_0^2 - 2rr_0 \cos\psi + r^2}}$

$$G(r_i, r_{i+1}) = \int_{r_i}^{r_{i+1}} g(r)\,dr = G_0(r_{i+1}) - G_0(r_i)$$

$$G_0(x) = \mu_r(x) + r_0\left(3\cos^2\psi - 1\right)\ln\left(w(x)\right)$$

$$\mu_r(x) = \frac{x^2 \cos\psi + x r_0\left(1 - 6\cos^2\psi\right) + 3r_0^2 \cos\psi}{sq(x)}$$

The function $\mu_r(x)$ is tended to infinity when $\psi = 0$ and $x = r_0$.
Let the grid by variable r is uniform and a nodes of the grid include the edges of the body. If the points where the force components are calculated are situated at the middles of intervals between neighboring nodes then provided $\psi = 0$ we obtain

$$\mu_r(r_0 + \Delta r/2) - \mu_r(r_0 - \Delta r/2) = -6r_0$$

and function $G(r_i, r_{i+1})$ hasn't any singularities, except only logarithmic ones.

- **3.2 The component of force of attraction by coordinate** θ_0

- Denote $$f(r) = \rho(\varphi, \theta, r), \quad g(r) = \frac{\partial}{\partial\theta_0} \frac{r^2}{\sqrt{r_0^2 - 2rr_0\cos\psi + r^2}}$$

$$G(r_i, r_{i+1}) = \int_{r_i}^{r_{i+1}} g(r)\,dr = G_0(r_{i+1}) - G_0(r_i).$$

- Then $$G_0(x) = r_0 \cos'\psi \left[\mu_\theta(x) + 3r_0\cos\psi \ln(w(x))\right],$$

- where $$\cos'\psi = \frac{\partial\cos\psi}{\partial\theta_0}, \quad \mu_\theta(x) = \frac{x^2 - 6r_0x\cos\psi + 3r_0^2}{sq(x)} + \frac{r_0x\cos\psi - r_0^2 + r_0sq(x)}{sq(x)\sin^2\psi}$$

In the computation the first integral we multiply the function $G_0(x)$ on multiplier $\frac{\sin^2\psi}{\cos'\psi}$

Denote the calculated value of the first integral as $I_1(\theta,\theta_0)$.

- Denote $f(\theta)=I_1(\theta,\theta_0)$, $g(x)=\frac{\sin\theta\cos'\psi}{\sin^2\psi}$
- Then $\int\limits_{\theta_j}^{\theta_{j+1}}\frac{\sin\theta\cos'\psi}{\sin^2\psi}d\theta=\frac{1}{\left(1-\sin^2\theta_0\sin^2\varphi_1\right)}\left[\mu_\theta^{(1)}\left(\theta_j,\theta_{j+1}\right)+\mu_\theta^{(2)}\left(\theta_j,\theta_{j+1}\right)+\mu_\theta^{(3)}\left(\theta_j,\theta_{j+1}\right)\right],$
- where $\varphi_1=\varphi-\varphi_0$,

$$\mu_\theta^{(1)}\left(\theta_j,\theta_{j+1}\right)=-\sin\left(2\theta_0\right)\left|\sin\varphi_1\right|\cos\varphi_1\, v_1\left(\theta_j,\theta_{j+1}\right)$$

$$\mu_\theta^{(2)}\left(\theta_j,\theta_{j+1}\right)=0.5\sin\theta_0\left[1-\sin^2\varphi_1\left(1+\cos^2\theta_0\right)\right]v_2\left(\theta_j,\theta_{j+1}\right)$$

$$\mu_\theta^{(3)}\left(\theta_j,\theta_{j+1}\right)=\cos\theta_0\cos\varphi_1\left(1+\sin^2\theta_0\sin^2\varphi_1\right)\left(\theta_{j+1}-\theta_j\right)$$

$$v_1\left(\theta_j,\theta_{j+1}\right)=\left[arctg\frac{\left(1-\sin^2\theta_0\cos^2\varphi_1\right)\theta-0.5\sin\left(2\theta_0\right)\cos\varphi_1}{\sin\theta_0\left|\sin\varphi_1\right|}\right]_{tg\theta_j}^{tg\theta_{j+1}}$$

$$v_2\left(\theta_j,\theta_{j+1}\right)=\left[\ln\frac{\theta^2\left(1-\sin^2\theta_0\cos^2\varphi_1\right)-\theta\sin\left(2\theta_0\right)\cos\varphi_1+\sin^2\theta_0}{1+\theta^2}\right]_{tg\theta_j}^{tg\theta_{j+1}}$$

Then the integral $\int\limits_{\theta_j}^{\theta_{j+1}} \frac{\sin\theta \cos'\psi}{\sin^2\psi} d\theta$ and the function $v_2\left(\theta_j, \theta_{j+1}\right)$ have the only logarithmic singularity in point $\left(\varphi = \varphi_0, \theta_j = \theta_0\right)$

- **3.3 The component of force of attraction by coordinate** φ_0
- Denote $f\left(r\right) = \rho\left(\varphi, \theta, r\right)$, $g\left(r\right) = \frac{\partial}{\partial \varphi_0} \frac{r^2}{\sqrt{r_0^2 - 2rr_0 \cos\psi + r^2}}$.
- Then $G_0\left(x\right) = r_0 \sin\theta \sin\theta_0 \sin\left(\varphi - \varphi_0\right)\left[\mu_\varphi\left(x\right) + 3r_0 \cos\psi \ln\left(w\left(x\right)\right)\right]$,
- where $\mu_\varphi\left(x\right) = \frac{x^2 - 6r_0 x \cos\psi + 3r_0^2}{sq\left(x\right)} + \frac{r_0 x \cos\psi - r_0^2 + r_0 sq\left(x\right)}{sq\left(x\right)\sin^2\psi}$.
- In the computation of the first integral we multiply the function $G_0\left(x\right)$
- on multiplier $\frac{\sin^2\psi}{\sin\theta \sin\left(\varphi - \varphi_0\right)}$ and denote the calculated value of the first
- integral with the such multiplier as $I_2\left(\theta, \theta_0\right)$.

Denote $f(\theta) = I_2(\theta, \theta_0)$, $g(\theta) = \dfrac{\sin^2\theta \sin(\varphi - \varphi_0)}{\sin^2\psi}$.

- Then $$\sin(\varphi - \varphi_0)\int_{\theta_j}^{\theta_{j+1}} \frac{\sin^2\theta}{\sin^2\psi} d\theta = \frac{1}{\left(1 - \sin^2\theta_0 \sin^2\varphi_1\right)}\left[\mu_\varphi^{(1)}(\theta_j, \theta_{j+1}) + \mu_\varphi^{(2)}(\theta_j, \theta_{j+1}) + \mu_\varphi^{(3)}(\theta_j, \theta_{j+1})\right],$$
- where $\varphi_1 = \varphi - \varphi_0$,

$$\mu_\varphi^{(1)}(\theta_j, \theta_{j+1}) = \sin\theta_0\left(\cos^2\varphi_1 - \sin^2\varphi_1 \cos^2\theta_0\right) sign\left(\sin(\varphi - \varphi_0)\right) v_1(\theta_j, \theta_{j+1})$$

$$\mu_\varphi^{(2)}(\theta_j, \theta_{j+1}) = \sin\varphi_1 \sin\theta_0 \cos\theta_0 \cos\varphi_1 \, v_2(\theta_j, \theta_{j+1})$$

$$\mu_\varphi^{(3)}(\theta_j, \theta_{j+1}) = \sin\varphi_1\left(\cos^2\theta_0 - \sin^2\theta_0 \cos^2\varphi_1\right)\left(\theta_{j+1} - \theta_j\right)$$

- The integral $\sin(\varphi - \varphi_0)\displaystyle\int_{\theta_j}^{\theta_{j+1}} \frac{\sin^2\theta}{\sin^2\psi} d\theta$ and the function $v_2(\theta_j, \theta_{j+1})$
- have the only logarithmic singularity.

The analytical calculation of potential and force of attraction of an ellipsoid for special case of density.

- Let us consider an ellipsoid in Cartesian coordinates

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

with elliptical distribution of density. It means that it has a constant density value on surfaces of similar ellipsoids satisfying the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=k^2, \qquad k\in[0,1]$$

Then the density will be a function of the only one parameter k

$$\rho(M)=\rho(k)=\rho\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)$$

4.1 The potential of an ellipsoid

- The potential of an ellipsoid with the such distribution of density is [1]

$$U(M_0) = \pi abc \int_{\lambda}^{\infty} \frac{\chi(k^2)}{R(s)} ds$$

$$\chi(k^2) = \int_{k^2}^{1} \rho(\alpha) d\alpha$$

$$R(s) = \sqrt{(a^2+s)(b^2+s)(c^2+s)}$$

$$k^2 = \frac{x_0^2}{a^2+s} + \frac{y_0^2}{b^2+s} + \frac{z_0^2}{c^2+s}$$

$$\lambda = \begin{cases} 0, & M_0 \in T \\ \lambda_0, & M_0 \notin T \end{cases},$$

- and λ_0 satisfies the equation $\dfrac{x_0^2}{a^2+\lambda_0} + \dfrac{y_0^2}{b^2+\lambda_0} + \dfrac{z_0^2}{c^2+\lambda_0} = 1.$

Let us consider an elongated ellipsoid of rotation along axis $\overrightarrow{OZ}$ with largest semi-axis unit length and other semi-axises with lengths equal to γ.

Let
$$\rho(\alpha)=\frac{1}{(1+\alpha)^2}$$

Pass to spherical coordinates (φ,θ,r).

Then
$$U(M_0)=\frac{\pi\gamma^2}{2}\int_{\lambda}^{\infty}\upsilon(s)\,ds\ ,$$

$$\upsilon(s)=\frac{\left[\left(\gamma^2+s\right)(1+s)-(1+s)r_0^2\sin^2\theta_0-\left(\gamma^2+s\right)r_0^2\cos^2\theta_0\right]}{\left[\left(\gamma^2+s\right)(1+s)+(1+s)r_0^2\sin^2\theta_0+\left(\gamma^2+s\right)\cos^2\theta_0\right]}\frac{1}{\left(\gamma^2+s\right)\sqrt{1+s}}.$$

This integral can be calculated analytically

$$\frac{1}{\pi\gamma^2}U(\theta_0,r_0)=\frac{q_+(s_2)}{(s_1-s_2)}\left[1-\frac{r_0^2\sin^2\theta_0}{\gamma_S}\left(s_1+\gamma^2\right)-\frac{r_0^2\cos^2\theta_0}{(1+s_2)}\right]+$$
$$\frac{q_-(s_1)}{(s_1-s_2)}\left[-1+\frac{r_0^2\sin^2\theta_0}{\gamma_S}\left(s_2+\gamma^2\right)+\frac{r_0^2\cos^2\theta_0}{(1+s_1)}\right]+q_+\left(-\gamma^2\right)\frac{r_0^2\sin^2\theta_0}{\gamma_S}$$

where s_1 and s_2 are smallest and largest roots of parabola

$$p_1(s) = s^2 + s\left(1+\gamma^2+r_0^2\right)+\gamma^2+r_0^2\left(\sin^2\theta_0+\gamma^2\cos^2\theta_0\right)$$

$$s_1 \in (-\infty,-1] \qquad\qquad s_2 \in [-1,0)$$

$$\gamma_S = s_1 s_2 + \gamma^2\left(s_1+s_2\right)+\gamma^4$$

$$q_-(s) = \sqrt{-1-s}\left[\frac{\pi}{2} - arctg\left(\frac{\sqrt{1+\xi_0}}{\sqrt{-1-s}}\right)\right]$$

$$q_+(s) = \sqrt{1+s}\,\ln\left(\frac{\sqrt{1+\xi_0}-\sqrt{1+s}}{\sqrt{\xi_0-s}}\right)$$

ξ_0 is largest root of parabola

$$p_2(s) = s^2 + s\left(1+\gamma^2-r_0^2\right)+\gamma^2-r_0^2\left(\sin^2\theta_0+\gamma^2\cos^2\theta_0\right),$$

where λ_0 satisfies the equation

$$r_0^2\left(\frac{\sin^2\theta_0}{\gamma^2+\lambda_0}+\frac{\cos^2\theta_0}{1+\lambda_0}\right)=1.$$

4.2 The components of force of attraction of the ellipsoid

$$\frac{\partial U}{\partial x}=-2\pi abcx\int_{\lambda}^{\infty}\frac{\rho\left(k^2\right)ds}{\left(a^2+s\right)R(s)},\quad \frac{\partial U}{\partial y}=-2\pi abcy\int_{\lambda}^{\infty}\frac{\rho\left(k^2\right)ds}{\left(b^2+s\right)R(s)},\quad \frac{\partial U}{\partial z}=-2\pi abcz\int_{\lambda}^{\infty}\frac{\rho\left(k^2\right)ds}{\left(c^2+s\right)R(s)}$$

- Choose $\rho(\alpha)=\dfrac{1}{1+\alpha}$.
- Then

$$\frac{\partial U}{\partial r_0}=-2\pi\gamma^2 r_0\left[Q_1(\lambda,\gamma)\sin^2\theta_0+Q_2(\lambda,\gamma)\cos^2\theta_0\right]$$

$$\frac{\partial U}{\partial \theta_0}=-\pi\gamma^2 r_0^2\sin(2\theta_0)\left[Q_1(\lambda,\gamma)-Q_2(\lambda,\gamma)\right]$$

$$\frac{\partial U}{\partial \varphi_0}=0$$

$$Q_1(\lambda,\gamma)=\int_{\lambda}^{\infty}\frac{\sqrt{1+s}}{L(s)\left(\gamma^2+s\right)}ds,\qquad Q_2(\lambda,\gamma)=\int_{\lambda}^{\infty}\frac{1}{L(s)\sqrt{1+s}}ds$$

$$L(s)=\left(\gamma^2+s\right)(1+s)+r_0^2(1+s)\sin^2\theta_0+r_0^2\left(\gamma^2+s\right)\cos^2\theta_0$$

The integrals $Q_1(\lambda,\gamma)$ and $Q_2(\lambda,\gamma)$ can be calculated analytically :

$$Q_1(\lambda,\gamma)=\frac{2}{\gamma_S(s_1-s_2)}\left[\left(s_1+\gamma^2\right)q_+(s_2)-\left(s_2+\gamma^2\right)q_-(s_1)-(s_1-s_2)q_+\left(-\gamma^2\right)\right]$$

$$Q_2(\lambda,\gamma)=\frac{2}{(s_1-s_2)}\left[\frac{q_+(s_2)}{(1+s_2)}-\frac{q_-(s_1)}{(1+s_1)}\right].$$

- **Numerical experiments**

To calculate the integral (3) we use the uniform grid by coordinate r inside the body and uniform grids by other coordinates. All the grids are displaced on half step of integration with respect to edges of integration.

The potential of an ellipsoid of rotation is the function that does not depends from coordinate φ_0 . Therefore it is sufficiently to replace the variable $\varphi = t^2$ after integration of inner integral.

Denote as ε_{jk} the error of calculation in percents in point θ_j, r_k for calculated value of potential $\tilde{U}\left(\theta_j, r_k\right)$ with respect to precise value $U\left(\theta_j, r_k\right)$ in the same node

$$\varepsilon_{jk} = 100 \times \left| 1 - \frac{\tilde{U}\left(\theta_j, r_k\right)}{U\left(\theta_j, r_k\right)} \right|$$

The values of average $\varepsilon_{av} = \frac{1}{N_r N_\theta} \sum_{k=1}^{N_r} \sum_{j=1}^{N_\theta} \varepsilon_{jk}$ and maximum $\varepsilon_{\max} = \max_{j,k} \varepsilon_{jk}$ errors are presented in table 1 .

The values of input parameters were: $N_\theta = N_r$, $N_\varphi = 100$, $\gamma = 0.5$, $r_0 \in [0.001,\ 10]$. The number of nodes on intervals $\left[0,\ R_{\theta_j}\right]$ and $\left[R_{\theta_j}, 10\right]$ was the same for every value θ_j .

N_r	50	100	200	400
ε_{av}	0.1331	0.0337	0.0090	0.0027
$\varepsilon_{\max}$	0.4835	0.1361	0.0379	0.0105

Table.1. The values of average and maximum errors of potential's calculation in percents for different number of nodes in grid .

To describe the error of calculation the components of the force of attraction we will choose the error in the following form:

$$\delta_{av}^{(r)} = \sum_{k=1}^{N_r}\sum_{j=1}^{N_\theta}\left[\frac{\partial \tilde{U}}{\partial r_0}\left(\theta_j, r_k\right) - \frac{\partial U}{\partial r_0}\left(\theta_j, r_k\right)\right]^2 \Bigg/ \sum_{k=1}^{N_r}\sum_{j=1}^{N_\theta}\left[\frac{\partial U}{\partial r_0}\left(\theta_j, r_k\right)\right]^2$$

$$\delta_{av}^{(\theta)} = \sum_{k=1}^{N_r}\sum_{j=1}^{N_\theta}\left[\frac{\partial \tilde{U}}{\partial \theta_0}\left(\theta_j, r_k\right) - \frac{\partial U}{\partial \theta_0}\left(\theta_j, r_k\right)\right]^2 \Bigg/ \sum_{k=1}^{N_r}\sum_{j=1}^{N_\theta}\left[\frac{\partial U}{\partial \theta_0}\left(\theta_j, r_k\right)\right]^2$$

$$\delta_{\max}^{(r)} = \max_{k,j}\left|\frac{\partial \tilde{U}}{\partial r_0}\left(\theta_j, r_k\right) - \frac{\partial U}{\partial r_0}\left(\theta_j, r_k\right)\right|$$

$$\delta_{\max}^{(\theta)} = \max_{k,j}\left|\frac{\partial \tilde{U}}{\partial \theta_0}\left(\theta_j, r_k\right) - \frac{\partial U}{\partial \theta_0}\left(\theta_j, r_k\right)\right|$$

N_r	50	100	200	400
$\delta_{av}^{(r)}$	0.726 E-5	0.715 E-6	0.852 E-7	0.139 E-7
$\delta_{\max}^{(r)}$	0.189 E-2	0.935 E-3	0.466 E-3	0.233 E-3
$\delta_{av}^{(\theta)}$	0.844 E-4	0.137 E-4	0.251 E-5	0.480 E-6
$\delta_{\max}^{(\theta)}$	0.928 E-3	0.450 E-3	0.220 E-3	0.108 E-3

In the calculation the component of the force of attraction by coordinate φ_0 we will consider the potential $U(\varphi_0, \theta_0, r_0)$ as a function of three variables .

Denote
$$\delta_{av}^{(\varphi)} = \sum_{i=1}^{N_\varphi} \sum_{k=1}^{N_r} \sum_{j=1}^{N_\theta} \left| \frac{\partial \tilde{U}}{\partial \varphi_0} (\varphi_i, \theta_j, r_k) \right|$$

$$\delta_{\max}^{(\varphi)} = \max_{k,j,i} \left| \frac{\partial \tilde{U}}{\partial \varphi_0} (\varphi_i, \theta_j, r_k) \right| .$$

Then $\delta_{av}^{(\varphi)} \approx 0.2 \times 10^{-10}$, $\delta_{\max}^{(\varphi)} \approx 0.4 \times 10^{-9}$ for all values N_r .

Thank you for attention !

Picture 1. The graph of the function $U(r,\theta_0)$ when $\theta_0 = 0$ (∘) and $\theta_0 = pi/2$ (•) .