Verified Computation of Hermitian (Symmetric) Solutions to Continuous-Time Algebraic Riccati Matrix Equations

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Outline

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The Riccati Equation

The matrix equation

$$R(X) := A^*X + XA - XSX + Q = 0, \qquad (1)$$

is called the continuous-time algebraic Riccati equation (CARE), where

$$\begin{aligned} A &\in \mathbb{C}^{n\times n}, \\ S = S^* &\in \mathbb{C}^{n\times n}, \\ Q = Q^* &\in \mathbb{C}^{n\times n}, \end{aligned}$$

are given and $X \in \mathbb{C}^{n\times n}$ is the unknown solution.

The Closed Loop Matrix

The matrix $A - SX$ is called the closed loop matrix associated with the CARE (1).

Stabilizing Solution of the CARE

- Several applications require a Hermitian positive semidefinite stabilizing solution of the CARE (1).
- A Hermitian solution X of (1) is a stabilizing solution if the closed loop matrix $A - SX$ is stable, i.e., the spectrum of $A - SX$ lies in the closed left half-plane.

Important Formula

vec-of-three-factors: $\mathrm{vec}(ABC) = (C^T \otimes A)\mathrm{vec}(B)$.

$\otimes$: Kronecker product of matrices
vec: stacks columns of a matrix into a long vector

Important Formula

notation for simplicity: „lowercase := vec(uppercase)"

$$b := \text{vec}(B).$$

Important Formula

so we write: $\mathrm{vec}(ABC) = (C^T \otimes A)\, b$

Fréchet Derivative of the function $R(X)$

The Fréchet derivative of R at X in the direction H is

$$R'(X) \cdot H = H(A - SX) + (A - SX)^* H,$$

which means that

$$\begin{aligned} r'(x) &= I \otimes (A - SX)^* + (A - SX)^T \otimes I \\ &\in \mathbb{C}^{n^2 \times n^2}. \end{aligned}$$

Outline

Enclosing Solutions to Riccati Matrix Equations

- Develop an efficient technique based on interval arithmetic which provides guaranteed error bounds for solutions of the continuous-time algebraic Riccati equation (1)

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An Interval Newton Method

Luther, Otten, Traczinski (1998) AND Luther, Otten (1999)

- The Fréchet derivative of R at X is used to derive an interval Sylvester matrix equation of the form $\boldsymbol{C}X + X\boldsymbol{D} = \boldsymbol{F}$,
- Transform the interval Sylvester equation into the large interval linear system $(I \otimes \boldsymbol{C} + \boldsymbol{D}^T \otimes I)x = \boldsymbol{f}$ with $x := \mathrm{vec}(X)$ and $\boldsymbol{f} := \mathrm{vec}(\boldsymbol{F})$ and solve it.

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Main Issue: Computational Complexity

The number of arithmetic operations needed to implement this interval Newton technique is roughly $\mathcal{O}(n^6)$!
because the coefficient matrix of the resulting interval linear system is $n^2 \times n^2$!

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Classical Krawczyk approach
Yano, Koga (2007) AND Yano, Koga (2008)

$$\boldsymbol{k}(\check{x}, \boldsymbol{x}) := \check{x} - R \cdot r(\check{x}) + \left(I_{n^2} - R \cdot r'(\boldsymbol{x})\right)(\boldsymbol{x} - \check{x}),$$

where

$$r : \mathbb{C}^{n^2} \to \mathbb{C}^{n^2}, \quad x \mapsto r(\check{x}) := \mathrm{vec}(R(\check{X})),$$

$$r'(x) = \left(I \otimes (A - SX)^* + (A - SX)^T \otimes I\right) \in \mathbb{C}^{n^2 \times n^2}.$$

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Main Issue Again: Computational Complexity

- Standard choice is to take $R \in \mathbb{C}^{n^2 \times n^2}$ as an approximate inverse of mid $\boldsymbol{r}'(\boldsymbol{x})$.
- R is needed explicitly. $I - R\,\boldsymbol{r}'(\boldsymbol{x})$ is also needed explicitly.
- Cost is $\mathcal{O}(n^5)$!
- The number of arithmetic operations needed to implement the classical Krawczyk approach is at-least $\mathcal{O}(n^5)$!

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Challenge

Reduce the cost to cubic !

The big question:

How to compute R and $I_{n^2} - R \cdot r'(\mathbf{x})$ more cheaply ?

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Essence of Krawczyk-Type Iterations

Theorem (Rump 1983, AND Frommer, H. 2009)

Assume that $f : D \subset \mathbb{C}^N \to \mathbb{C}^N$ is continuous in D. Let $\check{x} \in D$ and $\mathbf{z} \in \mathbb{IC}^N$ be such that $\check{x} + \mathbf{z} \subseteq D$. Moreover, assume that $\mathcal{P} \subset \mathbb{C}^{N\times N}$ is a set of matrices containing all slopes $P(\check{x}, y)$ for $y \in \check{x} + \mathbf{z} =: \mathbf{x}$. Finally, let $R \in \mathbb{C}^{N\times N}$. Denote $\mathcal{K}_f(\check{x}, R, \mathbf{z}, \mathcal{P})$ the set

$$\mathcal{K}_f(\check{x}, R, \mathbf{z}, \mathcal{P}) := \{-Rf(\check{x}) + (I - RP)z : P \in \mathcal{P}, z \in \mathbf{z}\}. \quad (2)$$

Then, if $\mathcal{K}_f(\check{x}, R, \mathbf{z}, \mathcal{P}) \subseteq \operatorname{int} \mathbf{z}$, the function f has a zero x^* in the set $\check{x} + \mathcal{K}_f(\check{x}, R, \mathbf{z}, \mathcal{P}) \subseteq \mathbf{x}$. Moreover, if $\mathcal{P}$ also contains all slope matrices $P(y, x)$ for the function f and for $x, y \in \mathbf{x}$, then this zero is unique in $\mathbf{x}$.

Slopes and Fréchet derivative of the function $R(X)$

Theorem

Assume that $\mathbf{X}$ is an Hermitian interval matrix and $X, Y \in \mathbf{X}$. Then, the interval arithmetic evaluation of the Fréchet derivative of R contains all its slopes.

Slopes and Fréchet derivative of the function $R(X)$

Proof.

Suppose that $X, Y \in \boldsymbol{X}$.

$$\begin{aligned} R(Y) - R(X) &= A^*Y + YA - YSY - A^*X - XA + XSX \\ &= A^*(Y - X) + (Y - X)A \\ &- \frac{1}{2}((Y + X)S(Y - X) + (Y - X)S(Y + X)), \end{aligned}$$

So,

$$\begin{aligned} r(y) - r(x) &= [I \otimes (A^* - \frac{1}{2}(Y + X)S) + \\ & (A^T - \frac{1}{2}(S(Y + X))^T) \otimes I](y - x). \end{aligned}$$

□

Slopes and Fréchet derivative of the function $R(X)$

Proof.

This means that

$$P(y,x) = I \otimes (A^* - \frac{1}{2}(Y+X)S) + (A^T - \frac{1}{2}(S(Y+X))^T) \otimes I.$$

Since $X, Y \in \mathbf{X}$, by the enclosure property of interval arithmetic we have

$$P(y,x) \in I \otimes (A^* - \mathbf{X}S) + (A - S\mathbf{X})^T \otimes I.$$

Since $\mathbf{X}$ is Hermitian, $X^*, Y^* \in \mathbf{X}$. Moreover, $S^* = S$. So, $A^* - \mathbf{X}S = (A - S\mathbf{X})^*$ and therefore

$$P(y,x) \in \underbrace{I \otimes (A - S\mathbf{X})^* + (A - S\mathbf{X})^T \otimes I}_{\text{interval arithmetic evaluation of } R'(X)}.$$

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So, What Do We Need ?

Step 1: An as thin as possible enclosure for

$$\mathcal{K}_f(\check{x}, R, \boldsymbol{z}, \mathcal{P}) := \{-R\, f(\check{x}) + (I - RP)z : P \in \mathcal{P}, z \in \boldsymbol{z}\}.$$

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Step 2: Check the relation $\mathcal{K}_f(\check{x}, R, \boldsymbol{z}, \mathcal{P}) \subseteq \operatorname{int} \boldsymbol{z}$.

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The Key: Spectral Decomposition of the Closed Loop Matrix

Let

$$A - SX = V\Lambda W \text{ with}$$

$$V, \Lambda, W \in \mathbb{C}^{n\times n},$$
$$VW = I,$$
$$\Lambda = Diag(\lambda_1, \lambda_2, \cdots, \lambda_n) \text{ diagonal.}$$

Consequence of the Spectral Decomposition

Recall: $r'(x) = I \otimes (A - SX)^* + (A - SX)^T \otimes I.$

$$r'(x) = (V^{-T} \otimes W^*) \cdot \left(I \otimes [W(A - SX)W^{-1}]^* + [V^{-1}(A - SX)V]^T \otimes I \right) \cdot (V^T \otimes W^{-*}),$$

Another basic formula: $(A \otimes B) \cdot (C \otimes D) = (A \cdot C \otimes B \cdot D)$.

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Consequence of the Spectral Decomposition

$$r'(x) = (V^{-T} \otimes W^*) \cdot \Big(I \otimes \underbrace{[W(A - SX)W^{-1}]^*}_{\simeq \Lambda} + \underbrace{[V^{-1}(A - SX)V]^T}_{\simeq \Lambda} \otimes I \Big) \cdot (V^T \otimes W^{-*}),$$

Consequence of the Spectral Decomposition: An Approximate Inverse for mid $r'(\boldsymbol{x})$

$$R = (V^{-T} \otimes W^{*}) \cdot \left(I \otimes \Lambda^{*} + \Lambda^{T} \otimes I\right)^{-1} \cdot (V^{T} \otimes W^{-*}),$$

Consequence of the Spectral Decomposition: An Approximate Inverse for mid $r'(\boldsymbol{x})$

$$R = (V^{-T} \otimes W^*) \cdot \Big(\underbrace{I \otimes \Lambda^* + \Lambda^T \otimes I} \Big)^{-1} \cdot (V^T \otimes W^{-*}),$$

***Extremely* important:**

$\Delta := I \otimes \Lambda^* + \Lambda^T \otimes I \; \in \mathbb{C}^{n^2 \times n^2}$ is diagonal.

Consequence of the Spectral Decomposition for the SECOND TERM in $\mathcal{K}_r(\check{x}, R, \boldsymbol{z}, \mathcal{P})$

Recall: $R = (V^{-T} \otimes W^*) \cdot \Delta^{-1} \cdot (V^T \otimes W^{-*})$.
We have

$$I_{n^2} - R \cdot r'(x) =$$

$$I_{n^2} - R(I_n \otimes (A - SX)^* + (A - SX)^T \otimes I_n) =$$

$$(V^{-T} \otimes W^*)\, \Delta^{-1}\, \Omega\, (V^T \otimes W^{-*}),$$

where
$\Omega = \Delta - I_n \otimes \left(W(A - SX)W^{-1}\right)^* - \left(V^{-1}(A - SX)V\right)^T \otimes I_n$.

Consequence of the Spectral Decomposition for the SECOND TERM in $\mathcal{K}_r(\check{x}, R, \boldsymbol{z}, \mathcal{P})$

Recall: $R = (V^{-T} \otimes W^*) \cdot \Delta^{-1} \cdot (V^T \otimes W^{-*})$.
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Outline

Alg. 1: Compute an Interval Matrix $\boldsymbol{Z}$ s.t. $\boldsymbol{z}$ Encloses the FIRST TERM $-R \cdot r(\check{x})$ with $R = (V^{-T} \otimes W^*) \cdot \Delta^{-1} \cdot (V^T \otimes W^{-*})$

1: Enclose $\boldsymbol{RES} := A^*\check{X} + \check{X}A - \check{X}S\check{X} + Q$.
2: Enclose $\boldsymbol{G} := \boldsymbol{I}_W^* \cdot \boldsymbol{RES} \cdot V$.
3: Enclose $\boldsymbol{H} := \boldsymbol{G}./D$.
4: Enclose $\boldsymbol{Z} := -W^*\boldsymbol{H}\boldsymbol{I}_V$.
5: Output $\boldsymbol{Z}$.

Cost of Alg. 1 is cubic.

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Alg. 2: Compute an Interval Matrix $\boldsymbol{U}$ s.t. $\boldsymbol{u}$ Encloses the Set of SECOND TERMS with x Replaced by $\check{x} + y$

Recall: $(I_{n^2} - R \cdot r'(\check{x} + y))y = (V^{-T} \otimes W^*)\, \Delta^{-1}\, \Omega\, (V^T \otimes W^{-*})y$,
where

$$\Omega = I_n \otimes \Lambda^* - I_n \otimes \left(W(A - S(\check{X} + Y))W^{-1} \right)^* +$$

$$\Lambda^T \otimes I_n - \left(V^{-1}(A - S(\check{X} + Y))V \right)^T \otimes I_n.$$

1: Enclose $\boldsymbol{ZZ} = \boldsymbol{I}_W^* \cdot \boldsymbol{Y} \cdot V$,
2: Enclose $\boldsymbol{P} = W \cdot (\, A - S \cdot (\check{X} + \boldsymbol{Y})\,) \cdot \boldsymbol{I}_W$.
3: Enclose $\boldsymbol{Q} = \boldsymbol{I}_V \cdot (\, A - S \cdot (\check{X} + \boldsymbol{Y})\,) \cdot V)$.
4: Enclose $\boldsymbol{E} = (\Lambda - \boldsymbol{P})^* \cdot \boldsymbol{ZZ} + \boldsymbol{ZZ} \cdot (\Lambda - \boldsymbol{Q})$.
5: Enclose $\boldsymbol{N} = \boldsymbol{E}./D$.
6: Enclose $\boldsymbol{U} = W^* \cdot \boldsymbol{N} \cdot \boldsymbol{I}_V$
7: Output $\boldsymbol{U}$.

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Main Algorithm

1: Use a floating point algorithm to get an approximate solution $\check{X}$ of the Riccati equation (1).
2: Use a floating point algorithm to compute V, W and Λ in the spectral decomposition of $A - S\check{X}$.
3: Put $D \in \mathbb{C}^{n\times n}$ s.t. column d_j of D is $\text{diag}(\overline{\Lambda}) + (\Lambda)_{jj}(1, \dots, 1)^T$.
4: Compute interval matrices $\boldsymbol{I}_W \ni W^{-1}$ and $\boldsymbol{I}_V \ni V^{-1}$.
5: Use Alg. 1 to compute an enclosure $\boldsymbol{Z}$ for $-R \cdot r(\check{x})$.
6: Put $\boldsymbol{X} = \boldsymbol{Z}$ and $k = 0$ {Prepare loop}
7: **repeat**
8:     Put $\boldsymbol{Y} = \square(0, \boldsymbol{X} \cdot [1-\varepsilon, 1+\varepsilon])$, increment k {ε-inflation}
9:     Use Alg. 2 to compute an interval matrix $\boldsymbol{U}$ such that $\boldsymbol{u}$ is an enclosure for the set $\{(I_{n^2} - R \cdot r'(\check{x} + y))y : \ y \in \boldsymbol{y}\}$
10:     Enclose $\boldsymbol{X} = \boldsymbol{Z} + \boldsymbol{U}$
11: **until** ($\boldsymbol{X} \subseteq \text{int}\, \boldsymbol{Y}$ or $k = 15$)
12: **if** $\boldsymbol{X} \subseteq \text{int}\, \boldsymbol{Y}$ **then** {successful termination}
13:     output $\boldsymbol{XX} = \check{X} + \boldsymbol{X}$
14: **end if**

Outline

Example 16 (well-conditioned) from benchmark examples for Riccati equations by Benner, Laub and Mehrmann, 1995
ARESOLV from Matlab's Robust Control Toolbox used for computing $\check{X}$

n	$\check{X}$	Our Algorithm double prec. res.		
	time	time	k	mrp arp
100	$2.3 \cdot 10^{-1}$	$7.0 \cdot 10^{-1}$	1	$4.0 \cdot 10^{-1}$ $8.8 \cdot 10^{-7}$
200	$4.6 \cdot 10^{-1}$	1.1	1	$8.4 \cdot 10^{-1}$ $1.9 \cdot 10^{-6}$
400	3.9	8.8	1	$8.4 \cdot 10^{-1}$ $3.0 \cdot 10^{-6}$
800	$3.0 \cdot 10^{+1}$	$6.4 \cdot 10^{+1}$	1	$9.6 \cdot 10^{-1}$ $2.9 \cdot 10^{-6}$

Source: Benchmark Examples for Riccati Equations by Benner, Laub and Mehrmann, 1995

Example 5: A 9th-order continuous state space model of a tabular ammonia reactor
$n = 9$, $t_{\check{X}} = 0.01$ sec., $t_{\mathbf{X}} = 0.05$ sec.
$mrp = 1.1 \times 10^{-12}$, $arp = 5.2 \times 10^{-14}$.

Example 19: A model of 35 coupled springs, dashpots and masses
$n = 140$, $t_{\check{X}} = 2.2$ sec.
$t_{\mathbf{X}} = 5.5$ sec. after 7 iterations
$mrp = 3.4 \times 10^{-9}$, $arp = 7.2 \times 10^{-13}$.

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Example 17: A feedback controller
$n = 21$, $t_{\check{X}} = 0.01$ sec.
Our algorithm fails because V is ill-conditioned
$\kappa_V = 2.4 \times 10^{+9}$,
condition number of Riccati equation: $\kappa_{Ricc} = 1.3 \times 10^{+9}$.

Summary

- Reduction of the cost for verification to cubic via spectral decomposition of the closed loop matrix $\Rightarrow$ comparable to the cost for getting $\check{X}$.
- Algorithm uses matrix-matrix operations $\Rightarrow$ fast in INTLAB.
- Algorithm will not succeed if the eigenvector matrix V is ill-conditioned.

- Outlook
  - Verify stabilizing property of a solution to the CARE (1)
  - Try recent algorithms for multiplication of interval matrices (by Rump & Ozaki, Ogita, Oishi & Nguyen, Revol and others)
  - Discrete-time Riccati equations

Questions ?

Comments ?

Or suggestions ?