Computing Reverse Interval Power Functions

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Outline

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Reverse Interval Operations

- Difficulties with inversion of interval power functions
  - Overvalue
  - Wrong

$x, y, z \in \mathbb{R}$, $x > 0$, $y \neq 0$

$$x^y = z$$
$$\Rightarrow x = z^{1/y}$$

Problem

$\mathbf{y} = [-0.5, 0.4], \mathbf{z} = [0.25, 0.25]$

$$\mathbf{z}^{1/\mathbf{y}} = \mathbf{z}^{]-\infty,+\infty[} = [0, +\infty[$$
$$\neq]0, \overline{z}^{1/\overline{y}}] \cup [\overline{z}^{1/\underline{y}}, +\infty[=]0, 0.03125] \cup [16, +\infty[$$

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Reverse Interval Operations

Definition

For a (partial) binary arithmetic operation $\circ$ there are two *binary reverse operations* on intervals,

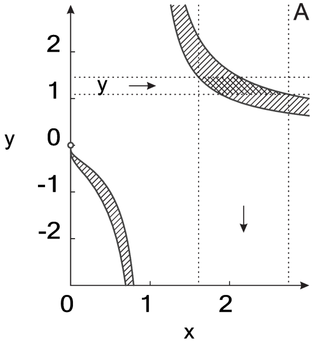
$$\circ_1^- : \overline{\mathbb{IR}} \times \overline{\mathbb{IR}} \to \wp(\mathbb{R})$$
$$\circ_2^- : \overline{\mathbb{IR}} \times \overline{\mathbb{IR}} \to \wp(\mathbb{R})$$

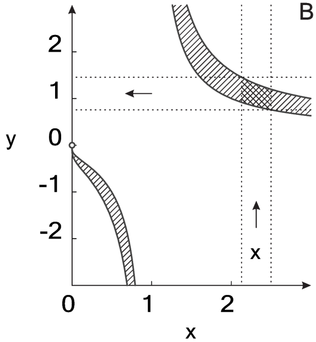
defined by

$$\circ_1^-(\mathbf{y}, \mathbf{z}) = \{x \in \mathbb{R} \mid \text{there exists } y \in \mathbf{y} \text{ with } x \circ y \in \mathbf{z}\}$$
$$\circ_2^-(\mathbf{x}, \mathbf{z}) = \{y \in \mathbb{R} \mid \text{there exists } x \in \mathbf{x} \text{ with } x \circ y \in \mathbf{z}\}$$

with $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \overline{\mathbb{IR}}$

Reverse Interval Operations





$$\circ_1^-(\mathbf{y},\mathbf{z}) = \{x \in \mathbb{R} \mid \text{there exists } y \in \mathbf{y} \text{ with } x \circ y \in \mathbf{z}\}$$
$$\circ_2^-(\mathbf{x},\mathbf{z}) = \{y \in \mathbb{R} \mid \text{there exists } x \in \mathbf{x} \text{ with } x \circ y \in \mathbf{z}\}$$

Reverse Interval Operations

Definition

For a (partial) binary arithmetic operation $\circ$ there are two *ternary reverse operations* on intervals,

$$\bullet_1^- : \overline{\mathbb{IR}} \times \overline{\mathbb{IR}} \times \overline{\mathbb{IR}} \to \overline{\mathbb{IR}}$$
$$\bullet_2^- : \overline{\mathbb{IR}} \times \overline{\mathbb{IR}} \times \overline{\mathbb{IR}} \to \overline{\mathbb{IR}}$$

defined by

$$\bullet_1^-(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathit{hull}(\{x \in \mathbf{x} \mid \text{there exists } y \in \mathbf{y} \text{ with } x \circ y \in \mathbf{z}\})$$
$$\bullet_2^-(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathit{hull}(\{y \in \mathbf{y} \mid \text{there exists } x \in \mathbf{x} \text{ with } x \circ y \in \mathbf{z}\})$$

with $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \overline{\mathbb{IR}}$

Reverse Interval Operations

$$\bullet_1^-(\mathbf{x},\mathbf{y},\mathbf{z}) = \mathit{hull}(\mathbf{x} \cap \circ_1^-(\mathbf{y},\mathbf{z}))$$
$$\bullet_2^-(\mathbf{x},\mathbf{y},\mathbf{z}) = \mathit{hull}(\mathbf{y} \cap \circ_2^-(\mathbf{x},\mathbf{z}))$$

$\Rightarrow$ It is sufficient to present reverse operations $\circ_1^-$ and $\circ_2^-$.

Reverse Interval Operations

$$\bullet_1^-(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathit{hull}(\mathbf{x} \cap \circ_1^-(\mathbf{y}, \mathbf{z}))$$
$$\bullet_2^-(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathit{hull}(\mathbf{y} \cap \circ_2^-(\mathbf{x}, \mathbf{z}))$$

$\Rightarrow$ It is sufficient to present reverse operations $\circ_1^-$ and $\circ_2^-$.

Name	Domain	Computation
pow1	$\mathbb{R}^+ \times \mathbb{R}$	$(x, y) \mapsto \exp(y \cdot \log x)$
pow2	$\mathbb{R}^+ \times \mathbb{R}$	$(x, y) \mapsto \exp(y \cdot \log x)$
	$\{0\} \times \mathbb{R}^+$	$(x, y) \mapsto 0$
	$\mathbb{R}^- \times \mathbb{Z}$	$(x, y) \mapsto \begin{cases} \exp(y \cdot \log \lvert x \rvert) & \text{if } y \text{ even} \\ -\exp(y \cdot \log \lvert x \rvert) & \text{if } y \text{ odd} \end{cases}$
pow3	$\mathbb{R}^+ \times \mathbb{R}$	$(x, y) \mapsto \exp(y \cdot \log x)$
	$\{0\} \times \mathbb{R}^+$	$(x, y) \mapsto 0$
	$\mathbb{R}^- \times D$	$(x, \frac{m}{n}) \mapsto \begin{cases} \exp(\frac{m}{n} \cdot \log \lvert x \rvert) & \text{if } m \text{ even and } n \text{ odd} \\ -\exp(\frac{m}{n} \cdot \log \lvert x \rvert) & \text{if } m \text{ odd and } n \text{ odd} \end{cases}$ with $D = \{\frac{m}{n} \mid m, n \in \mathbb{Z},\ n \text{ odd}\}$

Outline

Reverse pow1

$$\text{pow1} \quad \mathbb{R}^+ \times \mathbb{R} \qquad (x, y) \mapsto \exp(y \cdot \log x)$$

- Interval overlapping $\circledcirc$
- $\mathbf{z} \circledcirc [0, 1]$

$\Rightarrow$ Eight groups of images on real intervals $\mathbf{z}$

Reverse pow1

$$\text{pow1} \quad \mathbb{R}^+ \times \mathbb{R} \qquad (x, y) \mapsto \exp(y \cdot \log x)$$

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- $\mathbf{z} \circledcirc [0, 1]$

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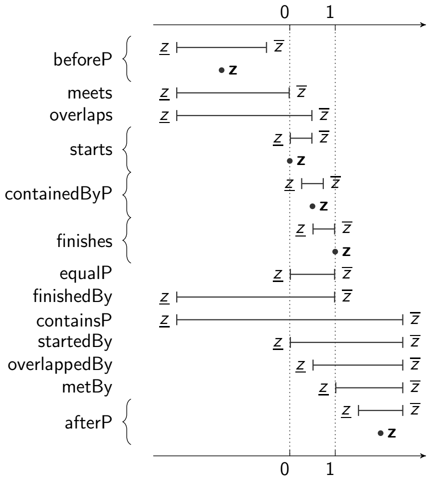
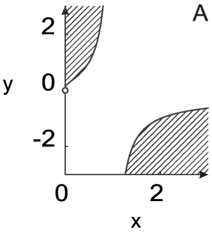
Interval overlapping

Sketches for the 13 states



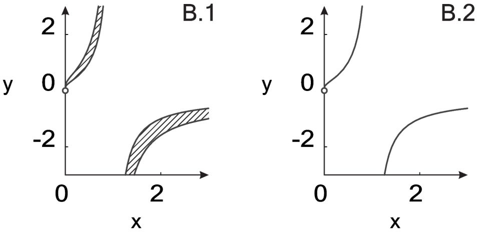
Overlapping states $\mathbf{z} \circledcirc [0, 1]$ with non-empty, bound $\mathbf{z} = [\underline{z}, \overline{z}]$.

Case A
Sketches



- z ⚭ [0, 1]
  - overlaps
  - starts

Case B
Sketches



- $\mathbf{z} \circledcirc [0, 1]$
  - containedByP

Look-up-table

$\mathrm{pow1}_1^-(\mathbf{y}, \mathbf{z})$:

	$\mathbf{y}$ ⚭ $[0,0]$					
$\mathbf{z}$ ⚭ $[0,1]$	beforeP	equalP	finishedBy	containsP	startedBy	afterP
overlaps/starts	$[\overline{z}^{1/\underline{y}}, +\infty[$	$\emptyset$	$[\overline{z}^{1/\underline{y}}, +\infty[$	$]0, \overline{z}^{1/\overline{y}}] \cup [\overline{z}^{1/\underline{y}}, +\infty[$	$]0, \overline{z}^{1/\overline{y}}]$	$]0, \overline{z}^{1/\overline{y}}]$
containedByP	$[\overline{z}^{1/\underline{y}}, \underline{z}^{1/\overline{y}}]$	$\emptyset$	$[\overline{z}^{1/\underline{y}}, +\infty[$	$]0, \overline{z}^{1/\overline{y}}] \cup [\overline{z}^{1/\underline{y}}, +\infty[$	$]0, \overline{z}^{1/\overline{y}}]$	$[\underline{z}^{1/\underline{y}}, \overline{z}^{1/\overline{y}}]$
finishes	$[1, \underline{z}^{1/\overline{y}}]$	$]0, +\infty[$	$]0, +\infty[$	$]0, +\infty[$	$]0, +\infty[$	$[\underline{z}^{1/\underline{y}}, 1]$
equalP/finishedBy	$[1, +\infty[$	$]0, +\infty[$	$]0, +\infty[$	$]0, +\infty[$	$]0, +\infty[$	$]0, 1]$
containsP/startedBy	$[\overline{z}^{1/\overline{y}}, +\infty[$	$]0, +\infty[$	$]0, +\infty[$	$]0, +\infty[$	$]0, +\infty[$	$]0, \overline{z}^{1/\underline{y}}]$
overlappedBy	$[\overline{z}^{1/\overline{y}}, \underline{z}^{1/\overline{y}}]$	$]0, +\infty[$	$]0, +\infty[$	$]0, +\infty[$	$]0, +\infty[$	$[\underline{z}^{1/\underline{y}}, \overline{z}^{1/\underline{y}}]$
metBy	$[\overline{z}^{1/\overline{y}}, 1]$	$]0, +\infty[$	$]0, +\infty[$	$]0, +\infty[$	$]0, +\infty[$	$[1, \overline{z}^{1/\underline{y}}]$
afterP	$[\overline{z}^{1/\overline{y}}, \underline{z}^{1/\underline{y}}]$	$\emptyset$	$]0, \underline{z}^{1/\underline{y}}]$	$]0, \underline{z}^{1/\underline{y}}] \cup [\underline{z}^{1/\overline{y}}, +\infty[$	$[\underline{z}^{1/\overline{y}}, +\infty[$	$[\underline{z}^{1/\overline{y}}, \overline{z}^{1/\underline{y}}]$

Note: For $\overline{z} \leq 0$ the result is $\emptyset$. Results for unbounded intervals can easily be obtained via limit values.

- $\mathrm{pow1}_2^-(\mathbf{x}, \mathbf{z})$ is defined in a similar manner
- Ternary function uses hull

Look-up-table

$\mathrm{pow1}_1^-(\mathbf{y}, \mathbf{z})$:

	$\mathbf{y} \circledcirc [0, 0]$					
$\mathbf{z} \circledcirc [0, 1]$	beforeP	equalP	finishedBy	containsP	startedBy	afterP
overlaps/starts	$[\overline{z}^{1/\underline{y}}, +\infty[$	$\emptyset$	$[\overline{z}^{1/\underline{y}}, +\infty[$	$]0, \overline{z}^{1/\overline{y}}] \cup [\overline{z}^{1/\underline{y}}, +\infty[$	$]0, \overline{z}^{1/\overline{y}}]$	$]0, \overline{z}^{1/\overline{y}}]$
containedByP	$[\overline{z}^{1/\underline{y}}, \underline{z}^{1/\overline{y}}]$	$\emptyset$	$[\overline{z}^{1/\underline{y}}, +\infty[$	$]0, \overline{z}^{1/\overline{y}}] \cup [\overline{z}^{1/\underline{y}}, +\infty[$	$]0, \overline{z}^{1/\overline{y}}]$	$[\underline{z}^{1/\underline{y}}, \overline{z}^{1/\overline{y}}]$
finishes	$[1, \underline{z}^{1/\overline{y}}]$	$]0, +\infty[$	$]0, +\infty[$	$]0, +\infty[$	$]0, +\infty[$	$[\underline{z}^{1/\underline{y}}, 1]$
equalP/finishedBy	$[1, +\infty[$	$]0, +\infty[$	$]0, +\infty[$	$]0, +\infty[$	$]0, +\infty[$	$]0, 1]$
containsP/startedBy	$[\overline{z}^{1/\overline{y}}, +\infty[$	$]0, +\infty[$	$]0, +\infty[$	$]0, +\infty[$	$]0, +\infty[$	$]0, \overline{z}^{1/\underline{y}}]$
overlappedBy	$[\overline{z}^{1/\overline{y}}, \underline{z}^{1/\overline{y}}]$	$]0, +\infty[$	$]0, +\infty[$	$]0, +\infty[$	$]0, +\infty[$	$[\underline{z}^{1/\underline{y}}, \overline{z}^{1/\underline{y}}]$
metBy	$[\overline{z}^{1/\overline{y}}, 1]$	$]0, +\infty[$	$]0, +\infty[$	$]0, +\infty[$	$]0, +\infty[$	$[1, \overline{z}^{1/\underline{y}}]$
afterP	$[\overline{z}^{1/\overline{y}}, \underline{z}^{1/\underline{y}}]$	$\emptyset$	$]0, \underline{z}^{1/\underline{y}}]$	$]0, \underline{z}^{1/\underline{y}}] \cup [\underline{z}^{1/\overline{y}}, +\infty[$	$[\underline{z}^{1/\overline{y}}, +\infty[$	$[\underline{z}^{1/\overline{y}}, \overline{z}^{1/\underline{y}}]$

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overlaps/starts	$[\overline{z}^{1/\underline{y}}, +\infty[$	$\emptyset$	$[\overline{z}^{1/\underline{y}}, +\infty[$	$]0, \overline{z}^{1/\overline{y}}] \cup [\overline{z}^{1/\underline{y}}, +\infty[$	$]0, \overline{z}^{1/\overline{y}}]$	$]0, \overline{z}^{1/\overline{y}}]$
containedByP	$[\overline{z}^{1/\underline{y}}, \underline{z}^{1/\overline{y}}]$	$\emptyset$	$[\overline{z}^{1/\underline{y}}, +\infty[$	$]0, \overline{z}^{1/\overline{y}}] \cup [\overline{z}^{1/\underline{y}}, +\infty[$	$]0, \overline{z}^{1/\overline{y}}]$	$[\underline{z}^{1/\underline{y}}, \overline{z}^{1/\overline{y}}]$
finishes	$[1, \underline{z}^{1/\overline{y}}]$	$]0, +\infty[$	$]0, +\infty[$	$]0, +\infty[$	$]0, +\infty[$	$[\underline{z}^{1/\underline{y}}, 1]$
equalP/finishedBy	$[1, +\infty[$	$]0, +\infty[$	$]0, +\infty[$	$]0, +\infty[$	$]0, +\infty[$	$]0, 1]$
containsP/startedBy	$[\overline{z}^{1/\overline{y}}, +\infty[$	$]0, +\infty[$	$]0, +\infty[$	$]0, +\infty[$	$]0, +\infty[$	$]0, \overline{z}^{1/\underline{y}}]$
overlappedBy	$[\overline{z}^{1/\overline{y}}, \underline{z}^{1/\overline{y}}]$	$]0, +\infty[$	$]0, +\infty[$	$]0, +\infty[$	$]0, +\infty[$	$[\underline{z}^{1/\underline{y}}, \overline{z}^{1/\underline{y}}]$
metBy	$[\overline{z}^{1/\overline{y}}, 1]$	$]0, +\infty[$	$]0, +\infty[$	$]0, +\infty[$	$]0, +\infty[$	$[1, \overline{z}^{1/\underline{y}}]$
afterP	$[\overline{z}^{1/\overline{y}}, \underline{z}^{1/\underline{y}}]$	$\emptyset$	$]0, \underline{z}^{1/\underline{y}}]$	$]0, \underline{z}^{1/\underline{y}}] \cup [\underline{z}^{1/\overline{y}}, +\infty[$	$[\underline{z}^{1/\overline{y}}, +\infty[$	$[\underline{z}^{1/\overline{y}}, \overline{z}^{1/\underline{y}}]$

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Outline

Reverse pow2

$$\text{pow2} \quad \begin{array}{ll} \mathbb{R}^+ \times \mathbb{R} & (x, y) \mapsto \exp(y \cdot \log x)) \\ \{0\} \times \mathbb{R}^+ & (x, y) \mapsto 0 \\ \mathbb{R}^- \times \mathbb{Z} & (x, y) \mapsto \begin{cases} \exp(y \cdot \log |x|) & \text{if } y \text{ even} \\ -\exp(y \cdot \log |x|) & \text{if } y \text{ odd} \end{cases} \end{array}$$

- Special interval overlapping $⚮$
  - $⚮ : ([\underline{a}, \overline{a}], [\underline{b}, \overline{b}]) \mapsto ([\underline{a}, \overline{a}] \mathbin{\text{⚭}} [\underline{b}, m], [\underline{a}, \overline{a}] \mathbin{\text{⚭}} [m, \overline{b}])$
  - 26 cases
- z $⚮ [-1, 1]$

$\Rightarrow$ Look-up-table

Reverse pow2

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- Special interval overlapping $\mathbin{\infty\!\!\!|}$
  - $\mathbin{\infty\!\!\!|} : ([\underline{a}, \overline{a}], [\underline{b}, \overline{b}]) \mapsto ([\underline{a}, \overline{a}] \mathbin{\infty} [\underline{b}, m], [\underline{a}, \overline{a}] \mathbin{\infty} [m, \overline{b}])$
  - 26 cases
- $\mathbf{z} \mathbin{\infty\!\!\!|} [-1, 1]$

$\Rightarrow$ Look-up-table

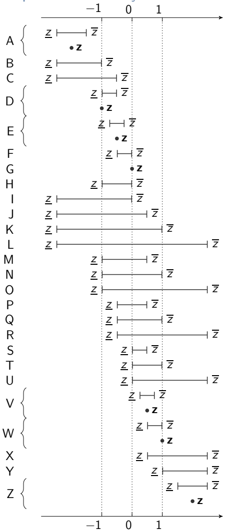
Special interval overlapping

Definition of the 26 states

$[\underline{a}, \overline{a}] \circledcirc\!\!\mid [\underline{b}, \overline{b}]$	Condition: $[\underline{a}, \overline{a}] \circledcirc [\underline{b}, m]$	Condition: $[\underline{a}, \overline{a}] \circledcirc [m, \overline{b}]$
A	beforeP	beforeP
B	meets	beforeP
C	overlaps	beforeP
D	starts	beforeP
E	containedByP	beforeP
F	finishes	meets
	⋮	
V	afterP	containedByP
W	afterP	finishes
X	afterP	overlappedBy
Y	afterP	metBy
Z	afterP	afterP

Special interval overlapping

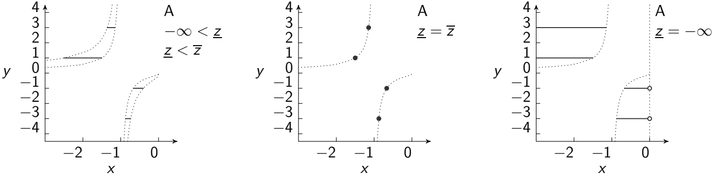
Sketches for the 26 states



Overlapping states $\mathbf{z} \circ\!\!|\!\!\circ [-1, 1]$ with non-empty, bound $\mathbf{z} = [\underline{z}, \overline{z}]$.

Case A

Sketches



$\mathbf{pow2}_1^-$

Require: $\mathbf{x}, \mathbf{y}, \mathbf{z}$

Ensure: $\mathbf{h} = \mathbf{pow2}_1^-(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \text{hull}(\mathbf{x} \cap \text{pow2}_1^-(\mathbf{y}, \mathbf{z}))$

1: $\mathbf{h}^+ \leftarrow \mathbf{pow1}_1^-(\mathbf{x}, \mathbf{y}, \mathbf{z})$

2: $\mathbf{x}^- \leftarrow \mathbf{x} \cap]-\infty, 0]$

3: $\mathbf{h}^- \leftarrow \text{hull}(\mathbf{x}^- \cap \text{pow2}_1^-(\mathbf{y}, \mathbf{z}))$ with Look-up-table

4: $\mathbf{h} \leftarrow \text{hull}(\mathbf{h}^- \cup \mathbf{h}^+)$

- $\mathbf{pow2}_2^-$ is defined in a similar manner

$\mathbf{pow2}_1^-$

Require: $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$

Ensure: $\mathbf{h} = \mathbf{pow2}_1^-(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \text{hull}(\mathbf{x} \cap \text{pow2}_1^-(\mathbf{y}, \mathbf{z}))$

1: $\mathbf{h}^+ \leftarrow \mathbf{pow1}_1^-(\mathbf{x}, \mathbf{y}, \mathbf{z})$

2: $\mathbf{x}^- \leftarrow \mathbf{x} \cap]-\infty, 0]$

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4: $\mathbf{h} \leftarrow \text{hull}(\mathbf{h}^- \cup \mathbf{h}^+)$

- $\mathbf{pow2}_2^-$ is defined in a similar manner

Difficult cases

- Some difficult cases for pow2_1^-
  - $\mathbf{z}$ oo|o $[-1, 1]$
    - A, E
    - V, Z
- Union of infinitely many disjoint intervals

$$\text{pow2}_1^-([1,+\infty],[2,3]) = \text{pow1}_1^-([1,+\infty],[2,3]) \cup \bigcup_{\substack{n\in[1,+\infty] \\ n \text{ even}}} [-3^{1/n}, -2^{1/n}]$$

$$\subseteq \text{pow1}_1^-([1,+\infty],[2,3]) \cup \underbrace{[-1.74,-1.41] \cup [-1.32,-1.18]}_{\text{disjoint}} \cup \ldots,$$

Problem

$$\text{pow2}_1^-(\mathbf{x},\mathbf{y},\mathbf{z}) = \mathit{hull}(\mathbf{x} \cap \text{pow2}_1^-(\mathbf{y},\mathbf{z}))$$

Difficult cases

- Some difficult cases for $\mathrm{pow2}_1^-$
  - $\mathbf{z}$ ⊚|○ $[-1, 1]$
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$$\subseteq \mathrm{pow1}_1^-([1, +\infty], [2, 3]) \cup \underbrace{[-1.74, -1.41] \cup [-1.32, -1.18]}_{\text{disjoint}} \cup \ldots,$$

Problem

$$\mathbf{pow2}_1^-(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathit{hull}(\mathbf{x} \cap \mathrm{pow2}_1^-(\mathbf{y}, \mathbf{z}))$$

Optimize $\mathbf{pow2}_1^-$ bounds

case Z

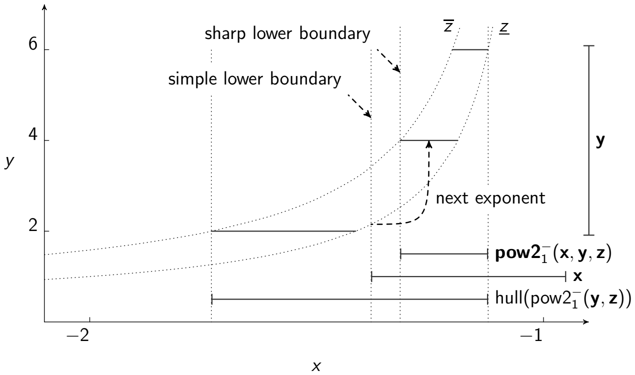
Require: $\mathbf{x} = [\underline{x}, \overline{x}]$ with $\overline{x} \leq 0$, $\mathbf{y} = [\underline{y}, \overline{y}]$ with $\underline{y} > 0$, $\mathbf{z} = [\underline{z}, \overline{z}]$ with $\underline{z} > 1$

Ensure: $[\underline{r}, \overline{r}] = \mathbf{pow2}_1^-(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{x} \cap \bigcup_{\substack{n \in \mathbf{y} \\ n \text{ even}}} [-|\overline{z}|^{1/n}, -|\underline{z}|^{1/n}]$

$[\underline{h}, \overline{h}] \leftarrow [-|\overline{z}|^{1/lee}, -|\underline{z}|^{1/gee}]$ {enclosure of union of intervals}
$[\underline{r}, \overline{r}] \leftarrow [\underline{x}, \overline{x}] \cap [\underline{h}, \overline{h}]$ {enclosure of result}
if $\underline{h} < \underline{x} < \overline{h}$ **then**
    {optimize left boundary of result}
    $a \leftarrow -\log_{|\underline{x}|} |\underline{z}|$
    $b \leftarrow \min\{y \in \mathbb{Z} \mid y \text{ even and } y \geq a\}$
    $c \leftarrow -|\overline{z}|^{1/b}$
    $\underline{r} \leftarrow \max\{\underline{r}, c\}$
end if
if $\underline{h} < \overline{x} < \overline{h}$ **then**
    {optimize right boundary of result}
    $a \leftarrow -\log_{|\overline{x}|} |\overline{z}|$
    $b \leftarrow \max\{y \in \mathbb{Z} \mid y \text{ even and } y \leq a\}$
    $c \leftarrow -|\underline{z}|^{1/b}$
    $\overline{r} \leftarrow \min\{\overline{r}, c\}$
end if
{if $\underline{r} > \overline{r}$, the result is the empty set}

Optimize $\mathbf{pow2}_1^-$ bounds

case Z



Outline

Reverse pow3

$$
\text{pow3} \quad
\begin{array}{ll}
\mathbb{R}^+ \times \mathbb{R} & (x, y) \mapsto \exp(y \cdot \log x) \\
\{0\} \times \mathbb{R}^+ & (x, y) \mapsto 0 \\
\mathbb{R}^- \times D & (x, \frac{m}{n}) \mapsto \begin{cases} |x|^{m/n} & \text{if } m \text{ even}, n \text{ odd} \\ -|x|^{m/n} & \text{if } m \text{ odd}, n \text{ odd} \end{cases}
\end{array}
$$

with $D = \{\frac{m}{n} \mid m, n \in \mathbb{Z},\ n \text{ odd}\}$

- Put together using reverse pow1

Reverse pow3

$$\text{pow3} \quad \begin{array}{ll} \mathbb{R}^+ \times \mathbb{R} & (x, y) \mapsto \exp(y \cdot \log x) \\ \{0\} \times \mathbb{R}^+ & (x, y) \mapsto 0 \\ \mathbb{R}^- \times D & (x, \frac{m}{n}) \mapsto \begin{cases} |x|^{m/n} & \text{if } m \text{ even, } n \text{ odd} \\ -|x|^{m/n} & \text{if } m \text{ odd, } n \text{ odd} \end{cases} \end{array}$$

with $D = \{\frac{m}{n} \mid m, n \in \mathbb{Z},\ n \text{ odd}\}$

- Put together using reverse pow1

$\mathrm{pow3}_1^-$

$$
\begin{aligned}
\mathrm{pow3}_1^-(\mathbf{y}, \mathbf{z}) = {} & \mathrm{pow1}_1^-(\mathbf{y}, \mathbf{z}) \\
& \cup \begin{cases} \{0\} & \text{if } \mathbf{y} \cap \mathbb{R}^+ \neq \emptyset \text{ and } 0 \in \mathbf{z} \\ \emptyset & \text{otherwise} \end{cases} \\
& \cup - \mathrm{pow1}_1^-(\mathbf{y}_{even}, \mathbf{z}) \\
& \cup - \mathrm{pow1}_1^-(\mathbf{y}_{odd}, -\mathbf{z}),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbb{Q}_{even} &= \{r \in \mathbb{Q} \mid r = \tfrac{m}{n} \text{ with } m \text{ even and } n \text{ odd}\}, \\
\mathbb{Q}_{odd} &= \{r \in \mathbb{Q} \mid r = \tfrac{m}{n} \text{ with } m \text{ odd and } n \text{ odd}\}, \\
\mathbf{y}_{even} &= \mathrm{hull}(\mathbf{y} \cap \mathbb{Q}_{even}) \\
\mathbf{y}_{odd} &= \mathrm{hull}(\mathbf{y} \cap \mathbb{Q}_{odd})
\end{aligned}
$$

pow3_2^-

$$
\begin{aligned}
\text{pow3}_2^-(\mathbf{x}, \mathbf{z}) = \text{pow1}_2^-(\mathbf{x}, \mathbf{z}) \\
\cup \begin{cases} \mathbb{R}^+ & \text{if } 0 \in \mathbf{x} \text{ and } 0 \in \mathbf{z} \\ \emptyset & \text{otherwise} \end{cases} \\
\cup \left(\text{pow1}_2^-(-\mathbf{x}, \mathbf{z}) \cap \mathbb{Q}_{even}\right) \\
\cup \left(\text{pow1}_2^-(-\mathbf{x}, -\mathbf{z}) \cap \mathbb{Q}_{odd}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbb{Q}_{even} &= \{r \in \mathbb{Q} \mid r = \tfrac{m}{n} \text{ with } m \text{ even and } n \text{ odd}\}, \\
\mathbb{Q}_{odd} &= \{r \in \mathbb{Q} \mid r = \tfrac{m}{n} \text{ with } m \text{ odd and } n \text{ odd}\},
\end{aligned}
$$

Outline

Summary

- Reverse interval power functions
- Definition
- Specification
- Implementation

Questions ?

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