On Inverse Problems for Degenerate Parabolic Equations with Many Spatial Variables

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Inverse problems

Our investigations deals with inverse problems of determination the unknown function p in non-divergent parabolic equation

$$\rho(t,x)u_t - a(t,x)\Delta u + \langle \vec{b}(t,x), u_x\rangle + c(t,x)u + \gamma(t)u = pg(t,x) + r(t,x),$$

$(t,x) \in Q \equiv [0,T] \times \bar{\Omega}$.

Inverse problem I (IP I).

The first inverse problem is following: It is required to define a pair of functions $\{u(t,x), p(x)\}$

$$\rho(t,x)u_t - \Delta u + \langle \vec{b}(t,x), u_x \rangle + c(t,x)u = p(x)g(t,x) + r(t,x), \tag{1}$$

$$u(0,x) = u_0(x), \quad x \in \bar{\Omega}, \tag{2}$$

$$u(t,x) = 0, \quad (t,x) \in [0,T] \times \partial\Omega, \tag{3}$$

$$\int_0^T u(t,x)\chi(t)\,dt = \varphi(x), \quad x \in \bar{\Omega}. \tag{4}$$

Inverse problem II (IP II).

The second inverse problem is following: It is required to define a pair of functions $\{u(t,x), p(t)\}$

$$u_t - a(t,x)\Delta u + \langle \vec{b}(t,x), u_x \rangle + c(t,x)u + \gamma(t)u = p(t)g(t,x) + r(t,x), \quad (5)$$

$$u(0,x) = u_0(x), \quad x \in \bar{\Omega}, \quad (6)$$

$$u(t,x) = 0, \quad (t,x) \in [0,T] \times \partial\Omega, \quad (7)$$

$$\int\limits_{\Omega} u(t,x)\chi(t)\,dx = \varphi(t), \quad t \in [0,T]. \quad (8)$$

It is assumed that the equations concerned are not uniformly parabolic. Namely, for the equation (1) we suppose that

$$0 \leq \rho(t, x) \leq \rho_1, \ \frac{1}{\rho(t, x)} \in L_q(Q), q > 1.$$

and for the equation (5) we suppose that

$$0 \leq a(t, x) \leq \rho_1, \ \frac{1}{a(t, x)} \in L_q(Q), q > 1.$$

Previous results

Inverse problems for nonuniformly (in particular, for degenerate) parabolic equations in the various posing earlier were considered in several papers of V.Isakov, I. Bouchouev, Lishang Jiang, Yourshan Tao, P. Cannarsa, J. Tort, M. Yamamoto, V.Kamynin, A.Kostin, A.Kozhanov, D.Lesnic, M.Hussein and many other mathematicians.

It should be noted that the direct and inverse problems for degenerate parabolic equations arises in various applied problems of hydrodynamics, climatology, problems of studying porous media, as well as in financial mathematics.

Method of investigation

Our method based on the estimates of the solution of corresponding direct problem with the constants computed explicitly.

Namely, the question of existence and the uniqueness of solutions of the inverse problem can be reduced to the question of the unique solvability of some operator equation

$$p = A(p)$$

in a certain Banach space and knowledge of these constants allows you to specify conditions on the input data of the inverse problem under which the operator considered is the contraction operator. Moreover we can give the explicit estimate for the unknown function p.

It is important that all the conditions of our theorems are issued in the form of easily verifiable inequalities.

Investigation of DP associated with IP I.

Let $p(x) \in L_2(\Omega)$ be known function. Put $f(t,x) = p(x)g(t,x) + r(t,x)$. Suppose that $f^2(t,x)/\rho(t,x) \in L_1(Q)$.

Definition 1. A generalized solution of the direct problem (1)–(3) is a function

$$u(t,x) \in C(0,T;L_1(\Omega)) \bigcap L_\infty(0,T;\overset{0}{W}{}_2^1(\Omega)) \bigcap W_s^{1,2}(Q), s > 1;$$

this function satisfies the equation (1) almost everywhere in Q, and the initial condition (2) in the norm of $C(0,T;L_1(\Omega))$.

General assumptions

All functions occurring in the input data of problem are measurable and satisfy the following conditions:

$$0 \le \rho(t,x) \le \rho_1, \rho(0,x) \le \rho_2, \rho(T,x) \le \rho_3,\ (t,x) \in Q;$$
$$1/\rho(t,x) \in L_q(Q), q > 1,\ \|1/\rho\|_{L_q(Q)} \le \rho_4; \qquad (A_1)$$

$$|\vec{b}(t,x)|^2/\rho(t,x),\ c^2(t,x)/\rho(t,x) \in L_\infty(Q),$$
$$|\vec{b}(t,x)|^2/\rho(t,x) \le K_{b,\rho},\ c^2(t,x)/\rho(t,x) \le K_{c,\rho},\ (t,x) \in Q; \qquad (B_1)$$
$$r^2(t,x)/rho(t,x) \in L_1(Q),\ g^2(t,x)/\rho(t,x) \in L_1(0,T;L_\infty(\Omega)),$$
$$\left\|r^2/\rho\right\|_{L_1(Q)} \le K_{r,\rho},\ \left\|g^2/\rho\right\|_{L_1(0,T;L_\infty(\Omega))} \le K_{g,\rho}; \qquad (C_1)$$

$$u_0(x) \in \overset{0}{W}{}_2^1(\Omega); \qquad (D_1)$$

$$\text{either } \rho_t \in L_1(Q),\ \rho_t \le 0,$$
$$\text{or } \rho_t^2(t,x)/\rho(t,x) \in L_\infty(Q),\ \left\|\rho_t^2/\rho\right\|_{L_\infty(Q)} \le K_\rho^*; \qquad (E_1)$$

Uniqueness theorem.

Theorem 1. Let conditions $(A_1) - (E_1)$ be satisfied. Then there is at most one generalized solution to the problem (1)–(3).

The proof is based on an energy estimate, but the nuance is that the higher derivatives of solutions only from $L_s(Q)$ and s can be less than 2.

Therefore, the proof must be carried out more carefully.

Existence theorem.

Theorem 2. Let conditions $(A_1)-(E_1)$ be satisfied. Let us put

$$q^* = \frac{2q}{q+1}(< 2) \text{ for } q \neq 3 \quad \text{and} \quad q^* = \frac{4}{3} \text{ for } q = 3, \tag{9}$$

$$\lambda^* = 3\left(K_{b,\rho} + \kappa^2(n,\Omega)K_{c,\rho}\right), \tag{10}$$

where $\kappa^2(n,\Omega)$ is a constant form Poincaré-Steklov inequality.

Then there exists generalized solution of the problem (1)–(3) for $s = q^*$ and we have the estimates

$$\sup_{0\le t\le T} \|u_x(t,\cdot)\|_2^2 + \|\rho u_t^2\|_{L_1(Q)} \le e^{\lambda^* T}\left(\|u_{0_x}\|_2^2 + 3\left\|\frac{f^2}{\rho}\right\|_{L_1(Q)}\right),$$

$$\|u_t\|_{L_{q^*}(Q)} \le \rho_4 e^{\lambda^* T}\left(\|u_{0_x}\|_2^2 + 3\left\|\frac{f^2}{\rho}\right\|_{L_1(Q)}\right),$$

$$\|u_{xx}\|_{L_{q^*}(Q)} \le C_1,$$

$$\|u(t_2,\cdot) - u(t_1,\cdot)\|_1 \le C_2|t_2 - t_1|^{(q^*-1)/q^*}.$$

Sketch of the proof.

Put $\rho_m(t,x) = \rho(t,x) + 1/m, m = 1, 2, \ldots$ and consider in Q the boundary problem for the equation

$$\rho_m(t,x)u_t^m - \Delta u^m + \sqrt{\rho_m(t,x)}\frac{(\vec{b}(t,x), u_x^m)}{\sqrt{\rho(t,x)}} + \sqrt{\rho_m(t,x)}\frac{c(t,x)}{\sqrt{\rho(t,x)}}u^m = = \sqrt{\rho_m(t,x)}\frac{f(t,x)}{\sqrt{\rho(t,x)}}, \quad (11)$$

with the boundary conditions (2),(3).

The equation (11) is uniformly parabolic with bounded coefficients and with the right hand side from $L_2(Q)$, the principal coefficient matrix is scalar.

Therefore, by virtue of results of O.Arena (1969), the solution $u^m(t,x)$ of the problem (11),(2),(3) exists and is unique in the space $L_\infty(0,T;\overset{0}{W}{}_2^1(\Omega)) \bigcap \bigcap W_2^{1,2}(Q)$.

Next we obtain estimates uniform with respect to m for the solution of the uniformly parabolic problem. Then we make the passage to the limit as $m \to \infty$, prove the existence of a solution to the original direct problem and the estimates.

Investigation of the IP I.

Assuptions

We suppose that conditions $(A_1) - (E_1)$ are satisfied.
Additionally assume that

$$\chi(t) \in L_\infty(0,T), (\rho\chi)_t \in L_2(0,T; L_\infty(\Omega)),\ |\chi(t)| \le K_\chi, t \in [0,T],$$

$$\left(\int_0^T \|(\rho\chi)_t(t,\cdot)\|_\infty^2\, dt\right)^{1/2} \le K_{\rho,\chi},\ \left|\int_0^T g(t,x)\chi(t)\, dt\right| \ge g_0 > 0, x \in \bar{\Omega}; \tag{F_1}$$

$$\varphi(x) \in W_2^2(\Omega) \bigcap \overset{0}{W}{}_2^1(\Omega),\ \|\Delta\varphi\|_2 \le K_\varphi. \tag{G_1}$$

Definition 2. A generalized solution of the inverse problem (1)–(4) is a pair of functions $\{u(t,x), p(x)\}$ that

$$u(t,x) \in C(0,T; L_1(\Omega)) \bigcap L_\infty(0,T; \overset{0}{W}{}_2^1(\Omega)) \bigcap W_s^{1,2}(Q), s > 1,\ p(x) \in Ł_2(\Omega);$$

these functions satisfy the equation (1) a.e. in Q, and function $u(t,x)$ satisfies the initial condition (2) in the norm of $C(0,T; L_1(\Omega))$ and overdetermination condition (4) a.e. in Ω.

Operator equation.

Let us derive an operator equation for finding the unknown function $p(x) \in L_2(\Omega)$. We multiply the equation (1) by $\chi(t)$ and integrate on $[0, T]$. Using overdetermination condition (4) we obtain

$$G(x)p(x) = \left[\rho(T,x)\chi(T)u(T,x) + \int_0^T (c\chi - (\rho\chi)_t)u\,dt + \int_0^T \langle\vec{b}, u_x\rangle\chi\,dt\right] - \\ - [\rho(0,x)\chi(0)u_0(x) + \Delta\varphi + R(x)]; \quad (12)$$

here

$$G(x) = \int_0^T g(t,x)\chi(t)\,dt, \quad R(x) = \int_0^T r(t,x)\chi(t)\,dt.$$

Denote

$$d(x) = -\frac{1}{G(x)}[\rho(0,x)\chi(0)u_0(x) + \Delta\varphi + R(x)].$$

By virtue of assumptions $(A) - (E), (F_1), (G_1)$

$$\|d\|_2 \leq K_d \equiv \frac{1}{g_0}[\rho_2|\chi(0)|\|u_0\|_2 + K_\varphi + K_\chi K_r T^{1/2}], \quad (13)$$

We introduce the operator $\mathcal{A}:\ L_2(\Omega) \to L_2(\Omega)$

$$\mathcal{A}(p) = \frac{1}{G(x)}\left[\rho(T,x)\chi(T)u(T,x) + \int_0^T (c\chi - (\rho\chi)_t)u\,dt + \int_0^T (\vec{b}, u_x)\chi\,dt\right] + d(x), \quad (14)$$

here $p(x)$ – is arbitrary function from $L_2(\Omega)$, and $u(t,x) \equiv u(t,x;p)$ – is a solution of the direct problem (1)–(3) with this $p(x)$ in the equation(1). Them the relation (12) can be written as operator equation

$$p = \mathcal{A}(p). \quad (15)$$

Equivalence of inverse problem and operator equation.

Lemma 1. Let conditions $(A_1) - (G_1)$ be satisfied. Then the operator equation (15) is equivalent to the inverse problem (1)–(4) in the following sense. If the pair $\{u(t,x),\ p(x)\}$ is a generalized solution of the inverse problem (1)–(4), then $p(x)$ satisfies the equation (15). Conversely, if $\hat{p}(x) \in L_2(\Omega)$ is a solution to the operator equation (15) and $\hat{u}(t,x)$ is a solution to the direct problem (1)–(3) with this function $\hat{p}(x)$ on the right part of the equation (1), then the pair $\{\hat{u}(t,x),\ \hat{p}(x)\}$ is a generalized solution of the inverse problem (1)–(4).

Existence of the solution of IP I.

Theorem 3. Let the conditions $(A_1) - (G_1)$ are satisfied, the constant q^* be defined in (9), the constant λ^* be defined in (10). Suppose that

$$\alpha \equiv \frac{(3K_{g,\rho})^{1/2}}{g_0} e^{\lambda^* T/2} \left[\kappa(n,\Omega)\left(|\chi(T)|\rho_3 + TK_cK_\chi + T^{1/2}K_{\rho,\chi}\right) + \right.$$
$$\left. + TK_bK_\chi\right] < 1. \quad (16)$$

Then there exists a generalized solution $\{u(t,x),\, p(x)\}$ of the inverse problem (1)–(4) with $u(t,x) \in W^{1,2}_{q^*}(Q)$. Moreover, such a solution is unique, and the following estimates hold:

$$\|p\|_2 \le \frac{e^{\lambda^* T/2}}{(1-\alpha)g_0} \left[\kappa(n,\Omega)\left(|\chi(T)|\rho_3 + TK_cK_\chi + T^{1/2}K_{\rho,\chi}\right) + TK_bK_\chi\right] \times$$
$$\times \left(\|u_{0_x}\|_2^2 + 3K_{r,\rho}\right)^{1/2} + \frac{1}{1-\alpha} K_d, \quad (17)$$

$$\sup_{0\le t\le T} \|u_x(t,\cdot)\|_2^2 \le e^{\lambda^* T}\left(\|u_{0_x}\|_2^2 + 6K_{\rho,\chi} + 6K_{g,\rho}\|p\|_2^2\right), \quad (18)$$

where the constant K_d is defined in (13).

We prove that due to (16), the operator $\mathcal{A}$ is contractive. So the equation (15) is uniquely solvable and can be solved by the iteration method, whence we obtain the estimate (17). And by virtue of the Lemma 1, the inverse problem is also uniquely solvable.

Example.

Consider in Q the inverse problem

$$(T-t)^{\theta} a(x) u_t - \Delta u = p(x), \quad (t,x) \in Q,$$

$$u(0,x) = u_0(x), \quad x \in \bar{\Omega}, \quad u(t,x) = 0, \quad (t,x) \in [0,T] \times \partial\Omega,$$

$$\frac{1}{T}\int_0^T u(t,x)\,dt = \varphi(x), \quad x \in \bar{\Omega}.$$

Here $\theta = \text{const} \in (1/2, 1)$, $a(x)$ – is arbitrary function which satisfies the condition $0 < a_1 \leq a(x) \leq a_2$, $\varphi(x)$ is an arbitrary function from $W_2^2(\Omega) \bigcap \bigcap \overset{0}{W}{}_2^1(\Omega)$, $u_0(x)$ is an arbitrary function from $\overset{0}{W}{}_2^1(\Omega)$.

The condition (16) will be written as

$$\left(\frac{3}{a_1(1-\theta)(2\theta-1)}\right)^{1/2} \theta a_2 \kappa(n,\Omega)\, T^{(\theta-1)/2} < 1. \tag{19}$$

Note that the constant $\kappa(n,\Omega)$ from the Poincar*é* condition tends to zero as $|\Omega| \to 0$. Therefore, the condition (19) is satisfied either for a sufficiently large T (and a fixed domain Ω), or if the size of the domain Ω is small (and T is fixed). In both these cases, the conditions of the theorem 3 are satisfied for our inverse problem , and, consequently, this inverse problem has a solution and, moreover, the only one .

Investigation of DP associated with IP II.

Let $p(t) \in L_2(0,T)$ be known function. Put $f(t,x) = p(t)g(t,x) + r(t,x)$. Suppose that $f^2(t,x)/\rho(t,x) \in L_1(Q)$.

Definition 3. A generalized solution of the direct problem (5)–(7) is a function

$$u(t,x) \in C(0,T;L_1(\Omega)) \bigcap L_\infty(0,T;\overset{0}{W}{}_2^1(\Omega)) \bigcap W_s^{1,2}(Q), s > 1;$$

this function satisfy the equation (5) almost everywhere in Q, and the initial condition (6) in the norm of $C(0,T;L_1(\Omega))$.

General assumptions

$$0 \le a(t,x) \le a_1,\ (t,x) \in Q; \quad 1/a(t,x) \in L_q(Q), q > 1,\ \|1/a\|_{L_q(Q)} \le a_2; \tag{A_2}$$

$$\frac{|\vec{b}(t,x)|^2}{a(t,x)},\ \frac{c^2(t,x)}{a(t,x)} \in L_\infty(Q),\ \gamma(t) \in L_2(0,T),$$

$$\frac{|\vec{b}(t,x)|^2}{a(t,x)} \le K_{b,a},\ \frac{c^2(t,x)}{a(t,x)} \le K_{c,a},\ (t,x) \in Q;\ \|\gamma\|_2 \le K_\gamma; \tag{B_2}$$

$$u_0(x) \in \overset{0}{W}{}_2^1(\Omega); \tag{C_2}$$

$$a_{x_k}(t,x) \in L_\infty(Q), k = 1,2,\dots,n, \quad \frac{|a_x(t,x)|^2}{a(t,x)} \in L_\infty(Q), \frac{|a_x(t,x)|^2}{a(t,x)} \le K_a,$$

$$\exists\, \Delta a \in L_1(0,T;L_\infty(\Omega)) \ \ \Delta a \le 0. \tag{D_2}$$

Uniqueness and existence theorems.

Theorem 4. Let conditions $(A_2)-(D_2)$ are satisfied. Then there is at most one generalized solution to the problem (5)–(7).

Theorem 5. Let conditions $(A_2)-(D_2)$ are satisfied. Let us put

$$q^* = \frac{2q}{q+1} \text{ for } q \neq 3 \quad \text{and} \quad q^* = \frac{4}{3} \text{ for } q = 3.$$

Then there exists a generalized solution $u(t,x)$ of the problem (5)–(7) with $s = q^*$. Moreover $\sqrt{a(t,x)}\Delta u \in L_2(Q)$ and we have the estimate

$$\sup_{0\le t\le \tau_0} \|u_x(t,\cdot)\|_2^2 \le 2\|u_{0_x}\|_2^2 + 6\|f^2/a\|_{L_1(Q_{\tau_0})}, \tag{20}$$

where τ_0 satisfies the relation

$$\frac{3}{2}\big(K_{b,a} + K_{c,a}\kappa^2(n,\Omega)\big)\tau_0 + \frac{\tau_0}{2} + K_\gamma \tau_0^{1/2} = \frac{1}{4},$$

and the estimates

$$\sup_{0\le t\le \tau} \|u_x(t,\cdot)\|_2^2 + \|a(\Delta u)^2\|_{L_1(Q_\tau)} \le C_1\left(\|u_{0_x}\|_2^2 + \|f^2/a\|_{L_1(Q_\tau)}\right),$$

$\forall\, \tau \in [0,T]$, C_1 does not depend on τ,

$$\|u_t\|_{L_2(Q)} \le C_2,\ \|u_{xx}\|_{L_{q^*}(Q)} \le C_3,$$

$$\|u(t_2,\cdot) - u(t_1,\cdot)\|_1 \le C_4|t_2 - t_1|^{(1/2},\ t_1, t_2 \in [0,T].$$

Remark. We note the importance of the estimate (20) in Theorem 5. Although it is local in time, it contains explicitly written constants on the right-hand side, which allows it to be effectively used in studies on the unique solvability of coefficient inverse problems for degenerate parabolic equations.

Investigation of the IP II.

Assuptions

We suppose that conditions $(A_2) - (D_2)$ are satisfied.
Additionally assume that

$$\frac{g^2(t,x)}{a(t,x)} \in L_\infty(0,T;L_1(\Omega)),\ \frac{r^2(t,x)}{a(t,x)} \in L_1(Q),$$

$$\|g^2/a\|_{L_\infty(0,T;L_1(\Omega))} \leq K_{g,a},\ \|r^2/a\|_{L_1(Q)} \leq K_{r,a}; \qquad (E_2)$$

$$a(t,x)\omega(x) \in L_\infty(0,T;\overset{0}{W}{}^1_2(\Omega)),\ \omega(x) \in W^1_2(\Omega),\ \sup_{0\leq t\leq T} \|(a(t,\cdot)\omega(\cdot))_x\|_2 \leq K_{a,\omega},$$

$$\|\omega\|_2 \leq K_\omega,\ \left|\int_\Omega g(t,x)\omega(x)\,dx\right| \geq g_0 > 0; \qquad (F_2)$$

$$\varphi(t) \in W^1_2(0,T),\ \varphi(0) = \int_\Omega u_0(x)\omega(x)\,dx. \qquad (G_2)$$

Definition 4. A generalized solution of the inverse problem (5)–(8) is a pair of functions $\{u(t,x), p(t)\}$ that

$$u(t,x) \in C(0,T;L_1(\Omega)) \bigcap L_\infty(0,T;\overset{0}{W}{}^1_2(\Omega)) \bigcap W^{1,2}_s(Q),\, s > 1,\ p(t) \in Ł_2(0,T);$$

these functions satisfy the equation (5) a.e. in Q, and function $u(t,x)$ satisfies the initial condition (6) in the norm of $C(0,T;L_1(\Omega))$ and overdetermination condition (8) in the classical sense.

Operator equation.

As in the case of IP I, we derive an operator equation for finding the unknown function $p(t) \in L_2(0, T)$.

We introduce the operator $\mathcal{B}: \, L_2(0,T) \to L_2(0,T)$ by the formula

$$\mathcal{B}(p)(t) = \frac{1}{G(t)} \left[\varphi'(t) + \gamma(t)\varphi(t) + R(t) + \int\limits_{\Omega} \langle (a\omega)_x, u_x \rangle \, dx + \right.$$
$$\left. + \int\limits_{\Omega} \langle \vec{b}\omega, u_x \rangle \, dx + \int\limits_{\Omega} c\omega \, u \, dx \right], \quad (21)$$

where $p(t)$ is an arbitrary function in $L_2(0,T)$, and $u(t,x) \equiv u(t,x;p)$ is a solution of the direct problem (5)–(7) with this $p(t)$ in the right hand side of the equation (5). Here

$$G(t) = \int\limits_{\Omega} g(t,x)\omega(x) \, dx, \quad R(t) = \int\limits_{\Omega} r(t,x)\omega(x) \, dx.$$

Then we prove that inverse problem (5)–(8) is equivalent (in the sense of Lemma 1) to the operator equation

$$p = \mathcal{B}(p). \quad (22)$$

Existence of the solution of IP II.

Theorem 6. Let the conditions $(A_2) - (G_2)$ are satisfied. Then the generalized solution of the inverse problem (5)–(8) exists and is unique.

To prove this theorem we establish that some power of the operator $\mathcal{B}$ defined by the formula (21) is a contraction in space $L_2(0,T)$. So the operator equation (22) has a unique solution in $L_2(0,T)$.

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