Обратные задачи в механике композитных материалов

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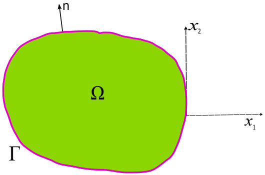
Linear elasticity problem Find $\boldsymbol{u} = (u_1, u_2)$, $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, such that

$$-\operatorname{div}\boldsymbol{\sigma} = \boldsymbol{f},\ \ \boldsymbol{\sigma} = \boldsymbol{D}\varepsilon(\boldsymbol{u}) \text{ in } \Omega, \tag{1}$$

$$\boldsymbol{\sigma}\boldsymbol{n} = 0 \ \text{ on } \Gamma. \tag{2}$$

$$\int_{\Omega} \boldsymbol{f}\rho = 0 \ \ \forall \rho \in R(\Omega). \tag{3}$$

$$R(\Omega) = \{\rho = (\rho_1, \rho_2) \mid \rho(x) = (c^1, c^2) + c^0(x_2, -x_1),$$
$$x = (x_1, x_2) \in \Omega;\ c^0, c^1, c^2 \in \mathbb{R}\}.$$



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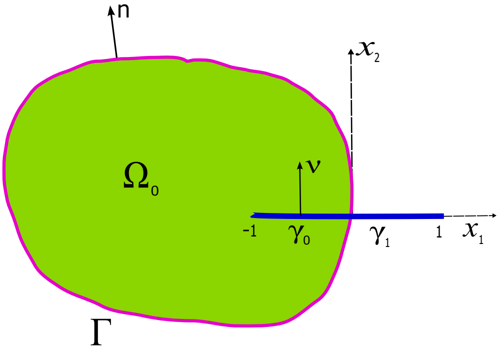
Geometry of the problem



Problem formulation Find $\boldsymbol{u} = (u_1, u_2)$, $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, defined in Ω_0, and v, w defined on γ such that

$$-\operatorname{div}\sigma = \boldsymbol{f},\ \ \sigma = D\varepsilon(\boldsymbol{u}) \text{ in } \Omega_0, \tag{4}$$

$$-(Av_{,1})_{,1} = [\sigma_\tau]\kappa_i,\ \ (Bw_{,11})_{,11} = [\sigma_\nu]\kappa_i \text{ on } \gamma_i,\ i = 0, 1, \tag{5}$$

$$\sigma n = 0 \ \text{ on } \Gamma;\ \ Av_{,1} = Bw_{,11} = (Bw_{,11})_{,1} = 0 \text{ as } x_1 = -1, 1, \tag{6}$$

$$u_\tau^- = v,\ \ u_\nu^- = w \text{ on } \gamma_0, \tag{7}$$

$$[u_\nu] \geq 0,\ \ \sigma_\nu^+ \leq 0,\ \ \sigma_\tau^+ = 0,\ \ \sigma_\nu^+[u_\nu] = 0 \text{ on } \ \gamma_0, \tag{8}$$

$$[v(0)] = [Av_{,1}(0)] = 0;\ \ [w(0)] = [w_{,1}(0)] = 0, \tag{9}$$

$$[Bw_{,11}(0)] = [(Bw_{,11})_{,1}(0)] = 0, \tag{10}$$

$$\int\limits_{\Omega_0} \boldsymbol{u} = 0,\ \int\limits_{\Omega_0} (u_{1,2} - u_{2,1}) = 0, \tag{11}$$

where $\kappa_0 = 1,\ \kappa_1 = 0,\ \sigma_\nu = \sigma_{ij}\nu_j\nu_i,\ \sigma_\tau = \sigma_{ij}\tau_j\tau_i,\ [q(0)] = q(0+) - q(0-)$.

Space

$$W = \{(u, v, w) \in H^1(\Omega_0)^2 \times H^1(\gamma) \times H^2(\gamma) \mid \quad u_\tau^- = v,\ u_\nu^- = w \text{ on } \gamma_0\}$$

and the subspace $W_1 \subset W$,

$$W_1 = \{(u, v, w) \in W \mid \int_{\Omega_0} u = 0, \int_{\Omega_0} (u_{1,2} - u_{2,1}) = 0,\ u = (u_1, u_2)\},$$

where $\tau = (\tau_1, \tau_2) = (\nu_2, -\nu_1)$, $u_\nu = u\nu$, $u_\tau = u\tau$. Spaces of infinitesimal rigid displacements

$$R(\Omega_0) = \{\rho = (\rho_1, \rho_2) \mid \rho(x) = (c^1, c^2) + c^0(x_2, -x_1),$$
$$x = (x_1, x_2) \in \Omega_0;\ c^0, c^1, c^2 \in \mathbb{R}\},$$

$$L(\gamma) = \{l = (l_1, l_2) \mid l_1(x_1) = d^2,\ l_2(x_1) = d^0 + d^1 x_1,$$
$$d^0, d^1, d^2 \in \mathbb{R}, ;\ x_1 \in (-1, 1)\}.$$

Inner product in the space $\boldsymbol{W}$,

$$\{(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}), (\bar{\boldsymbol{u}}, \bar{\boldsymbol{v}}, \bar{\boldsymbol{w}})\} = \int_{\Omega_0} u_i \int_{\Omega_0} \bar{u}_i + \int_{\Omega_0} u_{i,j}\bar{u}_{i,j} + \int_{\gamma} v_{,1}\bar{v}_{,1} + \int_{\gamma} w_{,11}\bar{w}_{,11}. \quad (12)$$

Proposition 1. *We have*

$$\boldsymbol{W} = \boldsymbol{W}_1 \oplus \boldsymbol{W}_1^{\perp},$$

where

$$\boldsymbol{W}_1^{\perp} = \{(\rho, l_1, l_2) \in \boldsymbol{W} \mid \rho \in \boldsymbol{R}(\Omega_0), \ (l_1, l_2) \in \boldsymbol{L}(\gamma)\}.$$

Energy functional $E: \ W \to \mathbb{R}$,

$$E(u,v,w) = \frac{1}{2}\int\limits_{\Omega_0} \sigma(u)\varepsilon(u) - \int\limits_{\Omega_0} fu + \frac{1}{2}\int\limits_{\gamma} Av_{,1}^2 + \frac{1}{2}\int\limits_{\gamma} Bw_{,11}^2.$$

Set of admissible displacements

$$P = \{(u,v,w) \in W \mid \ [u_\nu] \geq 0 \text{ on } \ \gamma_0\}.$$

Minimization problem

$$\inf_{(u,v,w)\in P\cap W_1} E(u,v,w). \tag{13}$$

$$(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \in P \cap W_1, \quad (14)$$

$$\int\limits_{\Omega_0} \sigma(\boldsymbol{u})\varepsilon(\bar{\boldsymbol{u}}-\boldsymbol{u}) - \int\limits_{\Omega_0} \boldsymbol{f}(\bar{\boldsymbol{u}}-\boldsymbol{u}) + \quad (15)$$

$$+\int\limits_{\gamma} A\boldsymbol{v}_{,1}(\bar{\boldsymbol{v}}_{,1}-\boldsymbol{v}_{,1}) + \int\limits_{\gamma} B\boldsymbol{w}_{,11}(\bar{\boldsymbol{w}}_{,11}-\boldsymbol{w}_{,11}) \geq 0 \quad \forall(\bar{\boldsymbol{u}},\bar{\boldsymbol{v}},\bar{\boldsymbol{w}}) \in P \cap W_1.$$

Proposition 2. *For any* $(\tilde{\boldsymbol{u}},\tilde{\boldsymbol{v}},\tilde{\boldsymbol{w}}) \in P$ *there exist* $(\bar{\boldsymbol{u}},\bar{\boldsymbol{v}},\bar{\boldsymbol{w}}) \in P \cap W_1$ *and* $(\rho, l_1, l_2) \in W_1^{\perp}$ *such that*

$$(\tilde{\boldsymbol{u}},\tilde{\boldsymbol{v}},\tilde{\boldsymbol{w}}) = (\bar{\boldsymbol{u}},\bar{\boldsymbol{v}},\bar{\boldsymbol{w}}) + (\rho, l_1, l_2).$$

Assume that the following condition holds

$$\int_{\Omega_0} f\rho = 0 \quad \forall \rho \in R(\Omega_0). \tag{16}$$

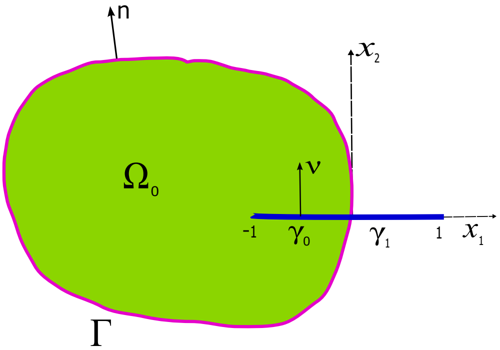
Variational formulation of the problem

$$(u, v, w) \in P \cap W_1, \tag{17}$$

$$\int_{\Omega_0} \sigma(u)\varepsilon(\tilde{u} - u) - \int_{\Omega_0} f(\tilde{u} - u) + \int_{\gamma} Av_{,1}(\tilde{v}_{,1} - v_{,1}) + \tag{18}$$

$$+ \int_{\gamma} Bw_{,11}(\tilde{w}_{,11} - w_{,11}) \geq 0 \quad \forall (\tilde{u}, \tilde{v}, \tilde{w}) \in P.$$

Theorem 1. There exists a solution of the problem (17)-(18) provided that the condition (16) holds.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Passage to limit with respect to rigidity parameter Find functions $u^\delta = (u_1^\delta, u_2^\delta)$, $\sigma = \{\sigma_{ij}\}, i, j = 1, 2$, defined in Ω_0, and v^δ, w^δ defined on γ such that

$$-\operatorname{div}\sigma = f, \ \sigma = D\varepsilon(u^\delta) \text{ in } \Omega_0, \tag{19}$$

$$-\delta^i(Av^\delta_{,1})_{,1} = [\sigma_\tau]\kappa_i, \ \delta^i(Bw^\delta_{,11})_{,11} = [\sigma_\nu]\kappa_i \text{ on } \gamma_i, \ i = 0, 1, \tag{20}$$

$$\sigma n = 0 \ \text{ on } \Gamma; \ Av^\delta_{,1} = Bw^\delta_{,11} = (Bw^\delta_{,11})_{,1} = 0 \text{ as } x_1 = -1, 1, \tag{21}$$

$$u_\tau^{\delta -} = v^\delta, \ u_\nu^{\delta -} = w^\delta \text{ on } \gamma_0, \tag{22}$$

$$[u_\nu^\delta] \ge 0, \ \sigma_\nu^+ \le 0, \ \sigma_\tau^+ = 0, \ \sigma_\nu^+[u_\nu^\delta] = 0 \text{ on } \ \gamma_0, \tag{23}$$

$$[v^\delta(0)] = 0, \ Av^\delta_{,1}(0-) = \delta Av^\delta_{,1}(0+); \ [w^\delta(0)] = [w^\delta_{,1}(0)] = 0, \tag{24}$$

$$Bw^\delta_{,11}(0-) = \delta Bw^\delta_{,11}(0+), \ (Bw^\delta_{,11})_{,1}(0-) = \delta(Bw^\delta_{,11})_{,1}(0+), \tag{25}$$

$$\int\limits_{\Omega_0} u^\delta = 0, \ \int\limits_{\Omega_0} (u^\delta_{1,2} - u^\delta_{2,1}) = 0, \tag{26}$$

where $\kappa_0 = 1, \ \kappa_1 = 0$.

Estimates uniform in δ,

$$\|\boldsymbol{u}^{\delta}\|_{H^1(\Omega_0)^2} \le \boldsymbol{c}, \tag{27}$$

$$\|\boldsymbol{v}^{\delta}\|_{H^1(\gamma_0)} + \|\boldsymbol{w}^{\delta}\|_{H^2(\gamma_0)} \le \boldsymbol{c}. \tag{28}$$

Estimate uniform for small δ,

$$\sqrt{\delta}\|\boldsymbol{v}^{\delta}\|_{H^1(\gamma_1)} + \sqrt{\delta}\|\boldsymbol{w}^{\delta}\|_{H^2(\gamma_1)} \le \boldsymbol{c}. \tag{29}$$

Assume that as $\delta \to 0$,

$$\boldsymbol{u}^{\delta} \to \boldsymbol{u} \text{ wealky in } H^1(\Omega_0)^2, \tag{30}$$

$$(\boldsymbol{v}^{\delta}, \boldsymbol{w}^{\delta}) \to (\boldsymbol{v}, \boldsymbol{w}) \text{ weakly in } H^1(\gamma_0) \times H^2(\gamma_0), \tag{31}$$

$$\sqrt{\delta}(\boldsymbol{v}^{\delta}, \boldsymbol{w}^{\delta}) \to (\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{w}}) \text{ weakly in } H^1(\gamma_1) \times H^2(\gamma_1). \tag{32}$$

Limit problem

Find functions $\boldsymbol{u} = (u_1, u_2)$, $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, defined in Ω_0, and v, w defined on γ_0 such that

$$-\operatorname{div}\sigma = \boldsymbol{f}, \quad \sigma = D\varepsilon(\boldsymbol{u}) \text{ in } \Omega_0, \tag{33}$$

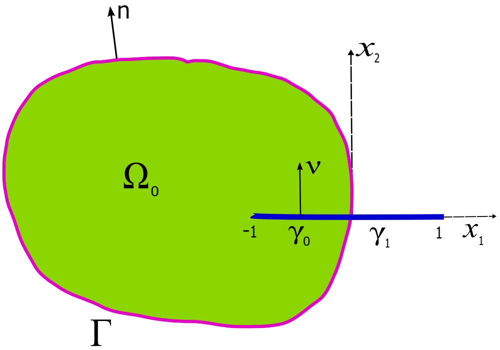
$$-(Av_{,1})_{,1} = [\sigma_\tau], \ (Bw_{,11})_{,11} = [\sigma_\nu] \text{ on } \gamma_0 \tag{34}$$

$$\sigma n = 0 \ \text{ on } \Gamma; \ Av_{,1} = Bw_{,11} = (Bw_{,11})_{,1} = 0 \text{ as } x_1 = -1, 0, \tag{35}$$

$$u_\tau^- = v, \ u_\nu^- = w \text{ on } \gamma_0, \tag{36}$$

$$[u_\nu] \geq 0, \ \sigma_\nu^+ \leq 0, \ \sigma_\tau^+ = 0, \ \sigma_\nu^+[u_\nu] = 0 \text{ on } \ \gamma_0, \tag{37}$$

$$\int\limits_{\Omega_0} \boldsymbol{u} = 0, \int\limits_{\Omega_0} (u_{1,2} - u_{2,1}) = 0. \tag{38}$$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Inverse problem

$$A(x_1) = A_g(x_1) = g_0 + g_1 x_1,\ B(x_1) = B_g(x_1) = g_2 + g_3 x_1;\ x_1 \in (-1, 1),$$

where $g = (g_0, g_1, g_2, g_3) \in \mathbb{R}^4$.
Let $G \subset \mathbb{R}^4$; $g = (g_0, g_1, g_2, g_3) \in G$ satisfies inequalities

$$c_0 \leq g_i \pm g_{i+1} \leq c_{00},\ i = 0, 2,$$

with positive constants c_0, c_{00}. For any $g \in G$ we can find the solution

$$(u^g, v^g, w^g) \in P \cap W_1, \tag{39}$$

$$\int_{\Omega_0} \sigma(u^g)\varepsilon(\bar{u} - u^g) - \int_{\Omega_0} f(\bar{u} - u^g) + \tag{40}$$

$$+ \int_{\gamma} \{A_g v^g_{,1}(\bar{v}_{,1} - v^g_{,1}) + B_g w^g_{,11}(\bar{w}_{,11} - w^g_{,11})\} \geq 0 \quad \forall (\bar{u}, \bar{v}, \bar{w}) \in P.$$

Problem formulation Find functions $\boldsymbol{u}^g = (u_1^g, u_2^g)$ defined in Ω_0 and functions v^g, w^g defined on γ, as well as $g \in G$ such that

$$(u^g, v^g, w^g) \in P \cap W_1, \tag{41}$$

$$\int_{\Omega_0} \sigma(u^g)\varepsilon(\bar{u} - u^g) - \int_{\Omega_0} f(\bar{u} - u^g) + \tag{42}$$

$$+ \int_{\gamma} \{A_g v_{,1}^g(\bar{v}_{,1} - v_{,1}^g) + B_g w_{,11}^g(\bar{w}_{,11} - w_{,11}^g)\} \geq 0 \quad \forall (\bar{u}, \bar{v}, \bar{w}) \in P,$$

$$w^g(x_0) = r. \tag{43}$$

Theorem 2. *There exist* $r_1, r_2 \in \mathbb{R}$, $r_1 \leq r_2$, *such that for any* $r \in [r_1, r_2]$ *the inverse problem* (41)-(43) *has a solution.*