Resource-intensive calculations in abstract algebra

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A large number of open problems in modern abstract algebra are related to finite objects that require heavy calculations either due to their large size or because of the algorithmic complexity. In this talk, we give striking examples from group theory of two unsolved problems that may benefit from the application of parallel-oriented computational methods.

Recall that a *group* is (roughly) a set of elements closed under a multiplication operation. There are many interesting examples of large finite groups that do not admit a compact realization suitable for computer calculations. One such example is the largest sporadic simple group [1], called *the Monster,* which has the astronomical order

$$808017424794512875886459904961710757005754368000000000$$

and whose smallest representation is a set of square matrices of size 196882 x 196882, each one occupying at least 4.8 G of memory. The multiplication of two such elements would be inefficient using even the fastest sequential algorithms. This is one of the reasons why the maximal subgroups of the Monster are still not completely described.

The set of words in a 2-element alphabet $\{x,y\}$ that have no subwords repeated 5 times, i. e. such that

$$w_1\, u\; u\; u\; u\; u\; w_2 = w_1\, w_2$$

is still not known to be finite or infinite. This question, being a particular case of the well-known Burnside's problem, may be stated in group-theoretic terms as whether or not the free 2-generator group B(2,5) of period 5 finite. The *conditional* Burnside group $B_0(2,5)$, which coincides with B(2,5) if the latter happens to be finite, has 5820766091346740722656251 elements. It is important but very difficult to calculate the order of this and other conditional Burnside groups.

Using calculations in the sequential system GAP [2], we determine the order of $B_0(2,3;12)$, the conditional free group of period 12 with elements not containing squares of x and cubes of y. In the talk, we briefly describe the algorithm used in these calculations and remark that the efficiency can be improved if the most resource-intensive part of the algorithm is implemented using the parallel architecture.

Conclusion. The investigation of some important open problems in group theory may benefit from the application of modern high-performance computational methods.

References

1. D. Gorenstein. Finite simple groups. An introduction to their classification // University Series in Mathematics. *Plenum Publishing Corp., New York,* 1982. x+333 pp.
2. The GAP Group. GAP - Groups, Algorithms, and Programming // Version 4.7.8; 2015. (http://www.gap-system.org)