Mathematical Model of Fluids Motion in Poroelastic Snow-Ice Cover

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Abstract

The problem of the dynamics of snow-ice cover is considered in the framework of the theory of poroelasticity. Snow-ice cover is considered as a three-phase medium consisting of water, air and ice. The mathematical model is based on the equations of conservation of mass for each phase, taking into account phase transitions, the equations of conservation of phase momenta in the form of Darcy's laws, the equation of conservation of momentum of the system as a whole, the rheological equation for porosity and the equation of heat balance of snow. In the full formulation, the temperature and pressure dependences of the liquid and air pressures are taken into account, as well as the temperature and viscosity coefficients of ice. The filtration of water in a thin poroelastic ice plate in the model case is considered. The solutions are obtained in quadratures.

Movement of air and water in a snow-ice cover

$$\frac{\partial(1-\phi)\rho_i}{\partial t} + div((1-\phi)\rho_i\vec{v_i}) = I_{wi} + I_{ai}, \quad \frac{\partial(\rho_w s_w\phi)}{\partial t} + div(\rho_w\phi s_w\vec{v_w}) = I_{iw} + I_{aw},$$

$$\frac{\partial(\rho_a s_a\phi)}{\partial t} + div(\rho_w\phi s_a\vec{v_w}) = I_{iw} + I_{wa}, \quad s_a + s_w = 1.$$

$$\phi(\vec{v}_w - \vec{v}_i) = -\frac{k(\phi)}{\mu_w(\theta)}(\nabla p_w - \rho_w\vec{g}), \quad \phi(\vec{v}_a - \vec{v}_i) = -\frac{k(\phi)}{\mu_a(\theta)}(\nabla p_a - \rho_a\vec{g}),$$

$$div\vec{v}_i = -\phi(\alpha(\theta)p_e + \beta(\theta)\frac{dp_e}{dt}), \quad \frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{v}_i\cdot\nabla),$$

$$p_{tot} = \phi p_f + (1-\phi)p_i, \quad p_e = (1-\phi)(p_i - p_f), p_f = s_w p_w + s_a p_a.$$

$$\nabla p_{tot} = \rho_{tot}\vec{g} + div((1-\phi)\eta(\theta)(\frac{\partial\vec{v}_i}{\partial\vec{x}} + \left(\frac{\partial\vec{v}_i}{\partial\vec{x}}\right)^*)),$$

$$\rho_{tot} = \phi\rho_f + (1-\phi)\rho_i, \rho_f = s_w\rho_w + s_a\rho_a,$$

$$(\rho_a c_a s_a\phi + \rho_w c_w s_w\phi + \rho_i c_i(1-\phi))\frac{\partial\theta}{\partial t} +$$
$$+(\rho_a c_a s_a\phi\vec{v}_a + \rho_w c_w s_w\phi\vec{v}_w + \rho_i c_i(1-\phi)\vec{v}_i)\nabla\theta =$$
$$= div\,(\lambda_c(\phi)\nabla\theta) + \nu\frac{\partial\rho_i(1-\phi)}{\partial t}.$$

Here $\rho_a, \rho_w, \rho_i, \vec{v}_a, \vec{v}_i, \vec{v}_w$ are the true densities and phase velocities (a is air, w is water, i is ice), is porosity, I_{lm} is the intensity of mass transfer from the l phase to the m phase in volume unit of the l phase per unit time, $I_{lm} = -I_{ml}$; $k(\phi)$ is the permeability, $\mu_w(\theta), \mu_a(\theta)$ are dynamic viscosities of water and air, $\vec{g}$ is the gravity acceleration vector; p_a and p_w are air and water pressure, $p_a - p_w = p_c(x, s_w)$ is capillary jump; $\alpha(\theta)$ and $\beta(\theta)$ are given environment parameters; p_{tot} is the total pressure, ρ_{tot} is the total density, $\eta(\theta)$ is the viscosity of the porous skeleton; θ is the temperature of the medium ($\theta_i = \theta_w = \theta$), $c_i = const > 0$, $c_w = const > 0$, $c_a = const > 0$ are heat capacity of ice, water and air at a constant volume, respectively; $\nu = const > 0$ is the specific heat of melting of ice; λ_{tot} is the thermal conductivity of the medium as a whole ($\lambda_{tot} = a_{tot} + b_{tot}\rho^2_{tot}$, $a_{tot} = const > 0$, $b_{tot} = const > 0$). To close the equations, it is necessary to take into account the dependence of the densities of liquid phases on pressures and temperature ($p = R\theta\rho$, R is the specific gas constant) or to set the densities constant.

J. A. D. Connolly, Y.Y. Podladchikov, Temperature-dependent viscoelastic compaction and compartmentalization in sedimentary basins, *Tectonophysics*, 2000.

J. Bear, Dynamics of Fluids in Porous Media, *Elsevier, New York*, (1972).

A.A. Papin, Y.Y. Podladchikov, Isothermal motion of two immiscible fluids in poroelastic medium, *Izvestiya Altai State University*, 2015. (in Russian)

The absence of air ($\rho_i, \rho_w = const$)

$$\frac{\partial(1-\phi)\rho_i}{\partial t} + div((1-\phi)\rho_i\vec{v_i}) = I_{wi}, \quad \frac{\partial(\rho_w s_w\phi)}{\partial t} + div(\rho_w\phi\vec{v_w}) = I_{iw},$$

$$\phi(\vec{v}_w - \vec{v}_i) = -\frac{k_w(\phi)}{\mu_w(\theta)}(\nabla p_w - \rho_w\vec{g}),$$

$$div\vec{v}_i = -\phi(\alpha(\theta)p_e + \beta(\theta)\frac{dp_e}{dt}), \quad \frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{v}_i\cdot\nabla),$$

$$p_{tot} = \phi p_w + (1-\phi)p_i, \quad p_e = (1-\phi)(p_i - p_w),$$

$$\nabla p_{tot} = \rho_{tot}\vec{g} + div((1-\phi)\eta(\theta)(\frac{\partial\vec{v}_i}{\partial\vec{x}} + \left(\frac{\partial\vec{v}_i}{\partial\vec{x}}\right)^*)), \quad \rho_{tot} = \phi\rho_w + (1-\phi)\rho_i,$$

$$\rho_w c_w\phi + \rho_i c_i(1-\phi))\frac{\partial\theta}{\partial t} + (\rho_w c_w\phi\vec{v}_w + \rho_i c_i(1-\phi)\vec{v}_i)\nabla\theta = div\,(\lambda_{tot}(\phi)\nabla\theta) +$$
$$+\nu\frac{\partial\rho_i(1-\phi)}{\partial t}.$$

The system is considered in the region $\Omega = (x, z) = [0, L] \times [0, H]$.

Small parameter

Let $\bar{x}, \bar{z}, \bar{t}$ are dimensionless variables defined by the equalities

$$\bar{x} = \frac{x}{L}, \quad \bar{z} = \frac{z}{H}, \quad \bar{t} = \varepsilon^k\tau_0 t, \quad \varepsilon = \frac{H}{L} \ll 1,$$

where $[L] = [H] = [\text{ m}]$, $[\tau_0] = [1/\text{ s}]$; k is a fixed parameter. We consider the most physical case $k = -2$:

$$\frac{\partial(1-\phi)}{\partial\bar{t}} + div((1-\phi)\bar{v}_i) = 0, \qquad \frac{\partial\phi}{\partial\bar{t}} + div(\phi\bar{v}_w) = 0,$$

$$\bar{v}^1_i = \bar{v}^1_w, \quad \frac{\tau_0\mu L^2}{kp}\phi(\bar{v}^2_w - \bar{v}^2_i) = -\phi^n\frac{\partial\bar{p}_w}{\partial\bar{z}},$$

$$\frac{\partial\bar{v}^1_i}{\partial x} + \frac{\partial\bar{v}^2_i}{\partial z} = -\beta p\frac{d(\bar{p}_{tot} - \bar{p}_w)}{d\bar{t}},$$

$$\frac{\partial}{\partial\bar{z}}\left((1-\phi)\frac{\partial\bar{v}^1_i}{\partial\bar{z}}\right) = 0, \quad 2\frac{\partial}{\partial\bar{z}}\left((1-\phi)\frac{\partial\bar{v}^2_i}{\partial\bar{z}}\right) + \frac{\partial}{\partial\bar{x}}\left((1-\phi)\frac{\partial\bar{v}^1_i}{\partial\bar{z}}\right) = 0,$$

$$(\frac{\rho_w c_w}{\rho_i c_i}\phi + (1-\phi))\frac{\partial\bar{\theta}}{\partial\bar{t}} + (\frac{\rho_w c_w}{\rho_i c_i}\phi\bar{v}^1_w + (1-\phi)\bar{v}^1_i)\frac{\partial\bar{\theta}}{\partial\bar{x}} + (\frac{\rho_w c_w}{\rho_i c_i}\phi\bar{v}^2_w + (1-\phi)\bar{v}^2_i)\frac{\partial\bar{\theta}}{\partial\bar{z}} =$$
$$= \frac{1}{L^2\tau_0 c_i}\frac{\partial}{\partial\bar{z}}\left((\frac{a}{\rho_i} + b(\phi(\frac{\rho_w}{\rho_i} - 1) + 1)^2)\frac{\partial\bar{\theta}}{\partial\bar{z}}\right) - \frac{\nu}{\delta c_i}\frac{\partial\phi}{\partial\bar{t}}.$$

J.Escher, M. Hillairet, P. Laurencot, C. Walker, Thin film equations with soluble surfactant and gravity: modeling and stability of steady states, *Mathematische Nachrichten*, 2012.

Solution in quadratures

$$v^1_i = v^1_w = A(x,t)\int_0^z \frac{1}{1-\phi}d\tau + B(x,t),$$

$$v^2_i = -1/2\frac{\partial A(x,t)}{\partial x}\int_0^z \frac{\tau}{1-\phi}d\tau + D(x,t)\int_0^z \frac{1}{1-\phi}d\tau + C(x,t),$$

$$v^2_w = -\frac{1}{\phi}\left[\int_0^z \frac{\partial}{\partial x}\left(A(x,t)\int_0^\xi \frac{1}{1-\phi}d\tau\right)d\xi - \frac{1}{2}\frac{\partial A(x,t)}{\partial x}\int_0^z \frac{\tau}{1-\phi}d\tau +\right.$$
$$\left.+\frac{\partial B(x,t)}{\partial x}z + D(x,t)\int_0^z \frac{1}{1-\phi}d\tau + E(x,t)\right] - \frac{1}{2}\frac{\partial A(x,t)}{\partial x}\int_0^z \frac{\tau}{1-\phi}d\tau +$$
$$+D(x,t)\int_0^z \frac{1}{1-\phi}d\tau + C(x,t),$$

$$p_w = \frac{\tau_0\mu L^2}{kp}\int_0^z \phi^{-n}\left[\int_0^\zeta \frac{\partial}{\partial x}\left(A(x,t)\int_0^\xi \frac{1}{1-\phi}d\tau\right)d\xi -\right.$$
$$\left.-\frac{1}{2}\frac{\partial A(x,t)}{\partial x}\int_0^\zeta \frac{\tau}{1-\phi}d\tau + \frac{\partial B(x,t)}{\partial x}\zeta + D(x,t)\int_0^\zeta \frac{1}{1-\phi}d\tau + E(x,t)\right]d\zeta + F(x,t).$$

Model case

As a special case, we consider the problem for the equations, supplemented by the following initial-boundary conditions:

$$\frac{\partial v^2_i}{\partial z}|_{z=0} = 0, \quad v^2_i|_{z=H} = C = const, \quad v^1_i|_{z=H} = B = const,$$
$$\frac{\partial v^1_i}{\partial z}|_{z=0} = 0, \quad \phi|_{t=0} = \phi^0(x,t), \quad \frac{\partial p_w}{\partial z}|_{z=H} = 0,$$
$$p_w|_{t=0} = p^0_w(x,z), p_i|_{t=0} = p^0_i(x,z), p_w|_{z=0} = p_0(x,t),$$
$$\theta|_{t=0} = \theta^0(x,z), \frac{\partial\theta}{\partial z}|_{z=0,z=H} = 0.$$

Then

$$v^1_i = v^1_w = B, \quad v^2_i = v^2_w = C,$$
$$\phi = \phi^0(x - Bt, z - Ct), \quad p_w = p_0(x,t).$$
$$p_i = p_0(x,t) - p^0_w(x - Bt, z - Ct) + p^0_i(x - Bt, z - Ct).$$

The equation for θ takes the form

$$(\frac{\rho_w c_w}{\rho_i c_i}\phi^0(x - Bt, z - Ct) + (1 - \phi^0(x - Bt, z - Ct)))\cdot$$
$$\cdot(\frac{\partial\theta}{\partial t} + B\frac{\partial\theta}{\partial x} + C\frac{\partial\theta}{\partial z}) =$$
$$= \frac{1}{L^2\tau_0 c_i}\frac{\partial}{\partial z}\left((\frac{a}{\rho_i} + b\rho_i(\phi^0(x - Bt, z - Ct)\frac{\rho_w}{\rho_i} + 1 - \phi^0(x - Bt, z - Ct)^2))\frac{\partial\theta}{\partial z}\right) -$$
$$-\frac{\nu}{\delta c_i}\frac{\partial\phi^0(x - Bt, z - Ct)}{\partial t}.$$

The equation for θ is a linear uniformly parabolic.