GROUP CLASSIFICATION OF EQUANIONS OF THE HYDROSTATIC MODEL FOR AN IDEAL FLUID IN LAGRANGE VARIABLES

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It is assumed that the pressure increases linearly with increasing depth: $p(x, y, z, t) = -gz + q(x, y, t)$. In this case, the equations have the form

$$\begin{aligned} &u_t + uu_x + vu_y + wu_z + q_x = 0, \qquad v_t + uv_x + vv_y + wv_z + q_y = 0, \\ &w_t + uw_x + vw_y + ww_z = 0, \qquad\quad u_x + y_y + w_z = 0. \end{aligned} \tag{1}$$

The basic Lie algebra of these equations was obtained in [Rodionov A. A. A Hydrostaic Model for an Ideal Fluid: Group Properties of Equations and their Solutions (SFU, Krasnoyarsk, 2015, no 8(3))] and has the form:

$$\begin{aligned} &\partial_t, \quad \partial_z, \quad t\partial_z + \partial_w, \quad t\partial_t + x\partial_x + y\partial_y + z\partial_z, \quad x\partial_x + y\partial_t + u\partial_u + v\partial_v + 2q\partial_q, \quad y\partial_x - x\partial_y + v\partial_u - u\partial_v, \\ &f_1(t)\partial_x + f_1'(t)\partial_u - xf_1''(t)\partial_q, \quad f_2(t)\partial_y + f_2'(t)\partial_v - yf_2''(t)\partial_q, \quad f_3(t)\partial_q \end{aligned}$$

with arbitrary smooth functions $f_j(t)$, $j = 1, 2, 3$.

If in the equations (1) we pass to the Lagrange variables ξ, η, ζ, then we obtain the equations

$$\begin{aligned} &x_\eta x_{t\zeta} - x_\xi x_{t\eta} + y_\eta y_{t\xi} - y_\xi y_{t\eta} = \omega(\xi, \eta, \zeta), \\ &(x_\xi y_\eta - x_\eta y_\xi)(1 + \omega_\zeta t) + (x_\zeta y_\xi - x_\xi y_\zeta)\omega_\eta t + (x_\eta y_\zeta - x_\zeta y_\xi)\omega_\xi t = 1, \end{aligned} \tag{2}$$

for functions $x(\xi, \eta, \zeta, t)$, $y(\xi, \eta, \zeta, t)$, $\omega(\xi, \eta, \zeta) = v_\xi - u_\eta$ is the projection of the vorticity vector onto the z axis at the initial instant of time.

Based on group analysis [Ovsjannikov L. V. Group Analysis of Differential Equations, Nauka, Moscow (1978)] for the equations (2), the group classification problem is set with respect to the function ω is posed.

It was proved that for an arbitrary choice of the function $\omega(\xi, \eta, \zeta)$, the equations (2) allow transformations of rotation in the plane (x, y) and a time shift in x, y with operators
$L_0 = \{y\partial_x - x\partial_y, \quad m(t)\partial_x, \quad n(t)\partial_y\}$.

The basic admissible operators are obtained when $\omega \equiv 0$, $\omega = const \neq 0$ and in the various cases when the derivatives $\partial\omega/\partial\xi, \partial\omega/\partial\eta, \partial\omega/\partial\zeta$ do or do not turn into a zero.

$\omega(\xi, \eta, \zeta)$	Operators
$\omega = const \neq 0$	$\varphi_\eta\partial_\xi - \varphi_\xi\partial_\eta$, $a_3\left(\xi\partial_\xi + \eta\partial_\eta + x\partial_x + y\partial_y\right)$, $a_2\left(-y\partial_x + x\partial_y\right)$, $\xi^3\partial_\zeta$, $a_0\partial_t$, $c_1\partial_x$, $c_2\partial_y$; $\varphi(\xi, \eta, \zeta)$, $a_0(\zeta)$, $a_2(\zeta)$, $a_3(\zeta)$, $\xi^3(\zeta)$, $c_1(\zeta, t)$, $c_2(\zeta, t)$ — arbitrary smooth functions
$\omega = 0$	$X = (\varphi_\eta + a_1\xi)\partial_\xi + (-\varphi_\xi + a_1\eta)\partial_\eta + \xi^3\partial_\zeta + \xi^4\partial_t + (a_1x - a_2y + c_1)\partial_x +$ $+a_2x + a_1y + c_2)\partial_y$; $\xi^4(\zeta, t) = 2\int a_1 dt - 2a_3t + a_4$, $\varphi(\xi, \eta, \zeta)$, $a_1(\zeta, t)$, $a_2(\zeta)$, $a_3(\zeta)$, $a_4(\zeta)$, $\xi^3(\zeta)$, $c_1(\zeta, t)$, $c_2(\zeta, t)$ — arbitrary smooth functions
$\omega_\xi \neq 0$, $\omega_\eta = \omega_\zeta = 0$ $\omega(\xi)$ — arbitrary function $\omega(\xi)$ — not arbitrary function	 $L_0^1 = \{y\partial_x - x\partial_y, \quad m(t)\partial_x, \quad n(t)\partial_y, \quad 2\eta\partial_\eta + x\partial_x + y\partial_y, \quad \partial_\zeta, \quad A_1(\xi, \zeta)\partial_\eta\}$ $\Big\{\omega/\omega_\xi\partial_\xi - \left(\omega/\omega_\xi\right)_\xi\eta\partial_\eta - t\partial_t, \quad y\partial_x - x\partial_y, \quad m(t)\partial_x, \quad n(t)\partial_y, \quad A_1(\xi, \zeta)\partial_\eta$ $-2(\omega/\omega_\xi)\partial_\xi + (2 + 2(\omega/\omega_\xi))_\xi\eta\partial_\eta + 2t\partial_t + x\partial_x + y\partial_y\}$, $m(t)$, $n(t)$, $A_1(\xi, \zeta)$ — arbitrary smooth functions
$\omega_\eta \neq 0$, $\omega_\xi = \omega_\zeta = 0$ $\omega(\eta)$ — arbitrary function $\omega(\eta)$ — not arbitrary function	 $L_0^2 = \{y\partial_x - x\partial_y, \quad m(t)\partial_x, \quad n(t)\partial_y, \quad 2\xi\partial_\xi + x\partial_x + y\partial_y, \quad \partial_\zeta, \quad A_2(\eta, \zeta)\partial_\xi\}$ $\Big\{-\left(\omega/\omega_\eta\right)_\eta\xi\partial_\xi + \omega/\omega_\eta\partial_\eta - t\partial_t, \quad y\partial_x - x\partial_y, \quad m(t)\partial_x, \quad n(t)\partial_y, \quad A_2(\eta, \zeta)\partial_\xi$ $(2 + 2(\omega/\omega_\eta))_\eta\xi\partial_\xi - 2(\omega/\omega_\eta)\partial_\eta + 2t\partial_t + x\partial_x + y\partial_y\}$, $m(t)$, $n(t)$, $A_2(\xi, \zeta)$ — arbitrary smooth functions
$\omega_\zeta \neq 0$, $\omega_\xi = \omega_\eta = 0$ $\omega(\zeta)$ — arbitrary function $\omega(\zeta)$ — not arbitrary function	 $L_0^3 = \{\varphi_\eta\partial_\xi - \varphi_\xi\partial_\eta, \quad a_3\left(\xi\partial_\xi + \eta\partial_\eta + x\partial_x + y\partial_y\right), \quad a_2\left(-y\partial_x + x\partial_y\right)$ $c_1\partial_x, \quad c_2\partial_y\}$ $\{\varphi_\eta\partial_\xi - \varphi_\xi\partial_\eta$, $b\left(\xi\partial_\xi + \eta\partial_\eta - (2\omega/\omega_\zeta)\partial_\zeta + 2\left[(1 + t\omega_\zeta)/\omega_\zeta + (t\omega\omega_{\zeta\zeta})/(\omega_\zeta)^2\right]\partial_t\right)$, $c_1\partial_x$, $c_2\partial_y$, $a_3(\omega/\omega_\zeta)\partial_\zeta - (t\omega\omega_{\zeta\zeta})/(\omega_\zeta)^2\partial_t, \quad a_1\left(-2\left[(1 + t\omega_\zeta)/\omega_\zeta\right]\partial_t + x\partial_x + y\partial_y\right), a_2\left(-y\partial_x + x\partial_y\right)\}$ $\varphi(\xi, \eta, \zeta)$, $a_1(\zeta, t)$, $a_2(\zeta)$, $b(\zeta)$, $\xi^3(\zeta)$, $c_1(\zeta, t)$, $c_2(\zeta, t)$ — arbitrary smooth functions

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